STUDENTS' CONCEPTIONS OF MATHEMATICS AS A DISCIPLINE

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Researchers have found that students' beliefs about mathematics impact the way in which they learn and approach mathematics in general. The purpose of this study is to categorize college students' various conceptions concerning mathematics as a discipline. Results from this study were used to create a preliminary framework for categorizing student conceptions. The results of this study indicate that the conceptions are numerous and range greatly in complexity. The results also suggest the need for further study to qualify the various student conceptions and the roles they play in students' understanding of and approach to performing mathematics.

Keywords: Students beliefs, mathematics,

Purpose and Background

College curriculums are designed to prepare students for specific careers. Part of this preparation is to enlighten the students about subjects that will play an integral part in their future. Thus, it is important to understand student perceptions of mathematics upon entering college. The purpose behind this investigation is to see how freshman calculus students thought of mathematics as a discipline. To clarify, this study is not focused on student beliefs pertaining to the nature of learning mathematics, but rather on what they feel mathematics is and the roles it plays.

Much research has been conducted on students' beliefs about mathematics concerning learning and motivation (e.g., Garafalo,1989; Hofer, 2001; Schoenfeld, 1985). These papers focused mainly on beliefs, which can be linked to academics. That is, the research focused on beliefs concerning the complexity of mathematics, ideas about learning and performing mathematics, and how these beliefs played a role in the learning process. Some student beliefs presented in these papers include math is hard, math is memorization of algorithms, and that there should be only one way to correctly answer a mathematics question.

Underhill (1988) summarized learners' beliefs about mathematics into four categories. These categories are *beliefs about mathematics as a discipline, beliefs about learning mathematics, beliefs about mathematics teaching,* and *beliefs about ones self within a social context in which mathematics teaching and learning can occur* (as cited by Op't Eynde, Corte & Verschaffel, 2002). Other research has shown that beliefs students have about mathematics play a role in the way the students approach learning and their motivations pertaining to mathematics (Hofer, 1999). Some beliefs are thought to be unhealthy and may even have a negative impact on learning mathematics (Spangler, 1992). The assessment of students' beliefs about mathematics can aid instructors in planning instruction and creating a classroom environment that can better help students develop a more enlightened system of beliefs about mathematics (NCTM, 1989).

Thompson (1984) investigated how teachers' perceptions of mathematics were related to their instructional behavior in the classroom. In her study, she aimed attention at how the teachers thought of mathematics as a field and how these views directed their instruction. As Thompson (1984) related teachers' beliefs about mathematics as a discipline and their beliefs about mathematics teaching, her research seems to be the closest, in the context of beliefs, to the current study. Two main differences are, that focus of this paper is on college students rather than instructors, and the beliefs being studied here pertain less to those about learning mathematics and more to what mathematics is and the roles it plays.

Participants and Methods

The data for this study come from semi-structured individual interviews (Bernard, 1988) conducted with undergraduate students at a large mid-atlantic university. Three of the four students interviewed were enrolled in integral calculus; the other was enrolled in differential calculus. All of the students were engineering majors. The students were selected on a volunteer basis. Three of the four students were expecting to attain an A- or better; the other was expecting a B. There were three parts to the interview. The first consisted of a sequence of short answer questions designed to determine the mathematical background of the interviewees. The second part of the interview consisted of three questions. The first asked the student to find a definite integral or evaluate the derivative (depending on the class in which the student was enrolled) of a polynomial. The second question engaged the students in a context novel to them, "fine functions" (Dahlberg, Housman;1997). The third question involved the students in an everyday situation where mathematics can be applied to solve the problem. The final part of the interview consisted of a series of questions asking the students to reflect on part two of the interview. The interview was designed to evoke the students' various beliefs of mathematics. Part two evoked these beliefs by directly engaging the students in mathematics and part three asked them to share their views of mathematics by reflecting on the questions from part two.

Grounded theory was used in completing the analysis for this study (Strauss & Corbin,1990). First, all of the responses were reviewed one question at a time. During this process similar conceptions emerged in the coding of the students' responses. After completing this iterative process, all of the similarly coded items were placed in respective groups and coded again. In this step, codes materialized with respect to how the students felt mathematics played a part in the various contexts (Layers). It was then determined the codes generalized across the various coded conceptions. The following section gives a background about the framework and some specifics are shared in the results section.

Development of the Framework

The model for this framework design was inspired by Zandieh's framework for analyzing students' understanding of derivative (2000). One must note that Zandieh's framework encompasses a broad range of understandings, which were developed by observing not only students at various levels but also by consulting textbooks, mathematicians, and mathematics education researchers. Here, however, the research is solely based on the student responses from this pilot study. No other additional sources played a role in the development of this framework. For this reason, there are no claims about this framework being exhaustive or in no need of refinement. However, much like Zandieh's framework, the framework presented here is not able to predict which beliefs mathematics students have and it does not place the various conceptions or layers in any sort of hierarchy. Thus, the framework provides no developmental nature for a student's beliefs of mathematics. As a result the framework is also not designed to make conjectures concerning what other beliefs may be a part of the student's overall view of mathematics based on the beliefs the student expressed. The framework is solely meant to arrange and format the various expressed student beliefs of mathematics.

The framework has been designed in the form of a matrix (Figure 1). The framework created for describing students' conceptions of mathematics as a discipline has two main elements: four concepts of mathematics and four layers of how these concepts can manifest themselves. A given student's conceptions of mathematics are organized as rows and the layers of each form columns. Each entry of the matrix represents the premise for that particular conception and layer combination.

Layer / Conception	Existence/Computation	Abstract	Justification
Numbers Operations Theorems	Mathematics is numbers, adding, subtracting and applying theorems to specific cases.	Each situation can be generalized, and math is applying theorems and computations to these situations.	Numbers, operations and theorems can be used to justify our solutions.
Tool for other sciences	Mathematics exists in other sciences. Mathematics is used by these other sciences.	Concepts from mathematics can be used as computations in general situations. For example the derivative can be used to explain velocity (but didn't give rise to it).	Use mathematics to verify that our answer from a certain science is correct.
As an interpretation	Mathematics appears in the world around us.	Specific mathematical ideas can be used to explain general ideas. Rate of change can be interpreted by derivatives (which gives rise to velocity for example).	Mathematics can be used to verify why these interpretations are correct.
Way of thinking	Mathematics is a way of thinking. Thinking through numbers and operations.	Mathematics can be applied to any situation.	The use of logical arguments results in a correct conclusion.

Figure 1: Framework for categorizing student conceptions about mathematics as a discipline.

Conceptions

Here the word *conception* is simply meant as something the student believes to be true. After analyzing the responses from all of the participants, four conceptions emerged from the data. These concepts include mathematics as: *numbers, operations and theorems; a tool for other sciences; a tool for interpretation;* and *a way of thinking.* All of these contexts were expressed verbally from the students except for most of *a way of thinking.* This context was interpreted based on the students' answers from part three and their written work from solving the various problems presented to them in part two.

The belief of *"Mathematics is about numbers, operations and theorems"* deals specifically with these three mathematical entities. Kloosterman (1996) noted that students often believe that math is simply computation. The students in this study presented this view; thus it is a conception in the framework. The category *mathematics is a tool for science* is based on the idea that mathematics is found in sciences such as physics and chemistry. The premise is that the science concepts existed before the mathematical concepts, which now serve to explain science. *Mathematics as a tool for interpretation* stems from the expressed belief that mathematics gives birth to the formulas used in different sciences. For example, because the derivative is used to calculate a rate of change (of anything), one can use the derivative to compute any rate of change that occurs in nature.

An effective example of *mathematics as a way of thinking* is demonstrated in the study by Carter and Norwood (1997) in which they found some teachers believe "to be good at mathematics you need a mathematical mind." This mathematical mind is thought of as a way of thinking. The documented belief from their study is used here only because it adequately exemplifies the conception of the phrase *a way of thinking* as it is used in this paper, further is was not used as a conception a priori. For clarification, it should be noted that *way of thinking* does not refer to characteristics of understanding, but rather a characteristic of mathematics.

Layers

The term *layers* simply refers to the way in which the belief of mathematics manifests itself in the student. To be clear, the term *layers* is not meant to imply any form of a hierarchical structure. *Layers* is also not meant to imply that any manifestation of each concept is more or less desirable than any other. Given the nature of this study, making these conjectures is beyond the scope of the research performed here. The framework developed here accounts for three layers in each context. The layers are: *existence/computation*, *abstraction*, and *justification*.

"Numbers, operations and theorems can be found in physics" and "Performing mathematical operations with certain theorems is an idea also used to solve physics problems" are examples of student responses that fall under the *existence/computation* layer of the *numbers, operations and theorems* context. When a student expresses one of the above conceptions in the context of generalized situations, then the student is said to have expressed that conception in the *abstraction* layer. The distinction between the existence/computation and abstraction layers is significant. In the former only numbers and operations are discussed, whereas the latter concerns using a certain set of operations or theorems in generalized situations, such as derivatives being used to calculate a rate of change.

A large number of people believe that mathematics is irrefutable and infallible (Cooney & Wiegel 2003). For this reason people think that mathematics is a sufficient way to show their answer to a problem (academic or not) is correct. This was evident in the analysis of the interviews conducted. The students, in various contexts expressed mathematics being used to justify an answer, and thus *justification* emerged as a layer. For example, a student who says, "using numbers and theorems to attain your solution will ensure the solution is correct", has expressed the conception of mathematics as *numbers, operations and theorems* in the *justification* layer.

Results

The following results are primarily centered around three students because their replies tend to best exemplify the concept-layer relationships in the framework. For brevity, only some of the analyses, which led to the framework are discussed in detail here.

Part two of the interview was designed to evoke the students' different beliefs of mathematics. While completing the first question in this part, it was clear the students were following an algorithm for solving the question. E.g. integrating the polynomial, then evaluating at the end points and finally subtracting. Upon being asked how they knew their answer was correct, all referenced a "rule" learned in class. When asked how they would explain the idea behind their work to someone with no knowledge of calculus, interesting results emerged. All of the students initially answered they would teach the rule referenced prior. When asked if there was a way they could explain it not using the rule, two responded they could not explain it without the rule. One attempted to relate area under the curve to the "reverse power rule" but quickly gave up and stated he would just teach the rule. The differential calculus student stated he would teach the limit definition of derivatives and then explain how it would give the formula for the slope of the line at any point. Based on their reliance upon using numbers and rules to explain their work to others, it may be inferred that the students think of mathematics in terms of numbers, operations, rules and potentially theorems. Within this conception, Cameron, when asked how he knew his answer was correct, replied that a rule was followed appropriately and the numbers and operations were all correctly applied. Combined with his later statement "If you can get results from your model that fit a theorem, then you know your answer is correct", gave rise to the *justification* layer of mathematics as numbers, operations and theorems.

When asked how the various questions from part two related to mathematics, Dan replied: "…that's what mathematics is, taking actual things like dollars, physical objects and turning it into a number that I can then manipulate with rules and theorems and things that have made sense from numbers so henceforth apply to actual things..." Here Dan expressed *mathematics as numbers, operations and theorems* in the *abstraction* layer when he stated math was taking "…physical objects and turning it into a number that I can manipulate with rules and theorems…" The creation of *mathematics as a tool for other sciences* as well as *mathematics as an interpretation* was also driven by Dan's responses. When asked if there was a benefit to learning mathematical concepts he went on to say that while attending high school, physics was only numbers and formulas to him and, "… where they would use calculus to explain something, it made me realize the calculus I was doing actually allowed me to understand it [physics]." Here Dan mentions the concept of a derivative is used to explain the ideas of velocity and acceleration from physics. It is important to note that Dan stated he was able to compute these values before but did not understand where they came from until explanation from calculus. He later added, "Algebra is plenty of numbers but calculus is making a form to this problem that works for all of nature which I find cool…" Here we also see Dan expressing *mathematics as interpretation* at the abstract layer; being able to take one mathematical idea and use it in numerous situations, "all of nature."

To the question "What do you feel mathematics is, as a discipline" Beth stated, "They [mathematicians] interpret other subject areas, for as an example physics… acceleration, velocity, you use calculus to interpret that stuff." The fact that she expressed the idea that the other science concepts existed first, and the mathematician interprets them computationally, led to the development of the abstract layer for mathematics as a tool for other sciences. The distinction between the two concepts, *mathematics as a tool for other sciences* and *as an interpretation* is slight, but significant. The former expresses the idea that mathematics is used after the fact, the latter expresses the idea of mathematics leading to the discovery of these scientific concepts such as velocity and acceleration.

It should be noted that while the framework was derived directly from the students' responses, two of the elements in the matrix are not accounted for in the results of the interview. These elements were placed in the framework for the sake of completeness and developing future work in this area. These elements are *mathematics as an interpretation* in the *justification layer* and *mathematics is a way of thinking* in the *abstract layer.*

Conclusion

The goal of this small study was to determine and then categorize the various student beliefs of mathematics as a discipline. Due to the size of this study, certainly not all student beliefs of mathematics have been discovered. However, there are significant implications based on the results. The results show that these concepts are numerous and range greatly in complexity.

Certainly the various beliefs affect how students approach, learn, are motivated by, and hence are interested in mathematics. For instance, students may be less motivated to indulge in mathematics based on their belief that mathematicians simply interpret the findings of other sciences. They may not see mathematics as being on the cutting edge of new science. On the other hand, if a student sees mathematics as yielding these scientific concepts and the scientists as simply applying them, the student may me more apt to joining the mathematical community. Also, the results here indicate that students do not reflect on mathematics as a problem solving tool, though their work indicates the use of these techniques. Often times we think of teaching mathematics as a tool for solving general problems. It is intriguing that, based on this study, students don't reflect on this. It is for these reasons, more work needs to be completed in this area of mathematics education.

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