STUDENTS' WAY OF THINKING ABOUT DERIVATIVE AND ITS CORRELATION TO THEIR WAYS OF SOLVING APPLIED PROBLEMS

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Previous researchers have examined students' understanding of derivative and their difficulties in solving applied problems and/or their difficulties in applying the basic knowledge of derivative in different contexts. There has not been much research approaching students' ways of thinking about derivative through the lens of applied questions. In this research, first I categorized the students' way of thinking about the basic concept of derivative by running a survey of questions addressing the different ways of thinking about derivative based on the existing research works. While analyzing these surveys, I used grounded theory and added more ways of thinking about derivative. I specially noticed very incomplete ways of thinking about derivative as described below. Since my goal was looking at the students' ways of thinking about derivative through the lenses of applied questions, I also piloted my applied questions survey with 51 multivariable calculus students. I noticed a lot of students struggling with defining variables (the initial translation as described below) and if they could define the variable, a lot of them struggled on applying their ways of thinking about derivative into solving the applied problem. These difficulties are great venues to study their ways of thinking about derivative using their struggle in the applied questions. This is a summary of my initial works on this ongoing research, the goal of which is to shed new insights into students' solving of applied problems.

Keywords: Derivative, Applied Problems, Ways of Thinking

Introduction

Derivative is one of the fundamental concepts covered in calculus. There is rich, extensive research on students' difficulties with the concept of derivative and their difficulties with applied derivative problems. However, not much is known about how students' conceptual understanding interacts with their work on applied problems. Sometimes students can think of derivative as the slope of a tangent line and sometimes as an instantaneous rate of change. They can even have incomplete ways of thinking about it for instance thinking of derivative as a slope of a function or thinking of it as rate of change. Students can also have combinations of these ways of thinking about derivative. Looking at correlations of students' different ways of thinking about derivative and how those understandings can impact their ability to solve applied problems can contribute to our understanding of undergraduate students' skills and conceptual understanding and generate insights into their thinking about application problems. Names used by researchers for these ways of thinking include "multiple representations," "contexts," or "layers of process object pairs" (Zandieh, 2000). Even our best students do not completely understand concepts taught in a course, and when faced with an unfamiliar problem, have difficulty solving the problem (Carlson, 1998; Selden et. al 1988; Selden et. al 2000; Bezuidenhout, 1996). The research question for this study is: Is there a relationship between students' multiple ways of thinking about derivative and their success in solving of applied problems?

Students' ways of thinking about derivative

Zandieh (2000) framed students' understanding of derivative in the multiple representations of graphical, verbal, physical, symbolical and other which were analyzed under the contexts of ratio, limit and function as the underlying concepts. Abboud and Habre (2006) used graphical, numerical, and symbolic views of derivative in assessing students' understanding of derivative. Kendal and Stacey (2003) used three representations of differentiation (graphical, symbolical, and physical) in creating what they called "Differentiation Competency Framework." In the application of these research works to the present study, I combined categories in some cases and added subcategories in others. This research uses the phrase *way of thinking* align with concept images or multiple representations of derivative which are based on constructivist cognitive theory.

Application Problems

In solving "real-world" problems, Tall (1991) wrote that the given problem is first translated from the context to the abstract level of calculus, the abstract problem is then solved, and the solution is translated back to the context. The first step obviously calls on students' conceptual knowledge of variables, algebra skills, and calculus concepts because it depends on the identification not only of the appropriate concepts in the given context but also of the relationships among them. The identification of appropriate concepts might involve the selection of one or more symbolized variables from among several concepts.

Research Design

Due to the complexity of the concept of derivative, we need to look at the students understanding of it from different perspectives. I used the cognitive variability and strategy choice as described by Stiegler (2003) as features of students thinking. Stiegler described how the students in different age use "multiple thinking strategies when solving problems of the same type" (P. 293). I describe these thinking strategies as ways of thinking about the concept of derivative. Therefore by identifying these multiple ways of thinking we can look at the students' problem solving strategies and its correlation to their problem solving strategies.

I used written surveys to collect data. Two separate surveys were created and administered to 125 differential calculus students and 51 multivariable calculus students at a large northeastern university. The first survey consisted of six questions three of which were used to look at students' fundamental ways of thinking about the derivative.

The second survey was focused on the applications of the derivative. This survey included one optimization problem where the students could either solve it intuitively or by just applying their basic understanding of derivative. The second question prompted the students to use derivative in solving a maximum/minimum question.

Some tasks came from existing research; others were created by the researcher. The first survey consisted of tasks addressing different possible "representations" or "concept images" held by the students. Most of these tasks were borrowed from existing research (Zandieh, 2000; Abboud & Habre, 2006; Kendal & Stacey, 2003; Carlson, 1998). The second survey was designed using tasks from White and Mitchelmore (1996).

Preliminary Data Analysis

I used the categories used by the other researchers and started analyzing the students surveys from differential calculus. While looking at the students responses there were a lot of responses where I could not fit their answers into any of the categories used by the other researchers. Using methods from Grounded Theory (Glaser & Strauss, 1990), I was able to add other categories in order to frame students' multiple ways of thinking about derivative.

The preliminary analysis was done based on what differential calculus students had written in response to the tasks on the surveys. Question one was based on Zandieh (2000).

1. You are talking with someone who just started high school. In a sentence or two explain to them what is meant by the derivative of a function. (Feel free to use any graph, symbols, or words in your explanation.)

Question one from firstthe survey

Question 2 was borrowed from Kendal and Stacey (2003) and was the question that the majority of their participants had the most difficulty with. Other research findings indicate that students don't define the derivative using the notion dy/dx and this question was designed to gather data on whether they would use it in their definitions.

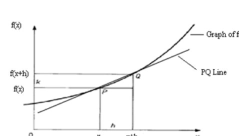
2. If y is a function of x, explain in words the meaning of the equation $\frac{dy}{dx} = 5$ when x = 10.

Question 2 from the first survey

It is known from previous research that likely not many students would use formal definition of derivative to express their ways of thinking therefore I added question 4 (Shown above) to the survey explicitly provided this as an opportunity for them to perhaps use it in their definition.

4. Here is a definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



Why does the formula give us the derivative of the function? Provide an explanation for your answer.

Question 4 from the first survey

Analysis of the remaining questions of first survey will be presented during the conference.

To analyze the second survey a couple of White & Mitchelmore's (1996) categories were used and other categories also emerged in the analysis. Table II is a sample of the analysis. As you can see this table includes four different categories of the students' difficulties with solving the applied problem. As explained earlier, these can be opportunities (venues) to explore the correlations of the students' multiple ways of thinking about derivative and their methods solving applied problems.

1. If the edge of a contracting cube is decreasing at a rate of 2 centimeters per minute, at what rate is the volume contracting when the volume of the cube is 64 cubic centimeters? (Provide an explanation for your answer.)

First question from the second survey

Results

The categories shown below are only based on what differential calculus students had written on their surveys. This provides one kind of window into their thinking but may not capture all of what they know. Some students might have used the same way of thinking several times in answering the survey questions however I counted them as once.
Symbolic category refers to the formal definition of derivative. Graphical ways of thinking refer to when the student uses slope of the tangent line in describing derivative. Incomplete Graphic refers to when students explain the derivative using only one term such as slope or only tangent line or when they say slope of a function. Numeric is referring to descriptions of the derivative using question 2 from the first survey. For instance when students define the derivative in terms of "derivative of y with respect to x is ten when x is 5, they are using a numerical example to describe the derivative which is why I used the title " numeric" for this group. Verbal refers to using Instantaneous rate of change in explaining derivative. Incomplete Verbal category refers to using just rate of change or rate of a function to explain the concept of derivative. Procedural is when the students talk about power rule or actually write an example of taking the derivative as a way of explaining it, for instance: f(x)=X^2 so f'(x)=2X. Category others refer to when students use area under the graph or accumulation to explain the derivative. For the purpose of this paper I used a very general term, "incomplete" in order to show that many students do not have a complete ways of thinking about derivative as we expect them do. Each one of this incomplete categories include several ways of thinking which are not complete in relation to the complete way of thinking about derivative as described by other research works or as defined in calculus books.
Ninety five out of 125 students had multiple ways of thinking about derivative however more than 70% of the students had incomplete ways of thinking about derivative as shown on the below table under "Incomplete Graphic", and "Incomplete Verbal" categories.

Table 1: Differential Calculus Students Surveys Results based on students answers to questions 1, 2 and 4

Categories of Student thinking*	Symbolic	Graphical	Incomplete Graphical	Numerical	Verbal	Incomplete Verbal	Physical In terms of Speed	Physical in terms of velocity/ Acceleration	Procedural	Others$
Number of Students thinking in this way	3 out 125	19 out of 125	91 out of 125	9 out of 125	18 out of 125	89 out of 125	2 out of 125	9 out of 125	31 out of 125	9 out of 125
%	2.4	15.2	72.8	7.2	14.4	71.2	1.6	7.2	24.8	7.2

40 out of the total of 51 students in multivariable calculus could not answer the question 1 from the second survey.

Table 2: Multivariable Calculus Students Surveys Results based on students answers to questions 1

Could not remember how to set it up, or did not try	Attempted but no symbolizing any variables for any quantities or wrong definitions of variables	Correct Variables but wrong modeling (the relationship between the variables)	Correct Translation but wrong calculus- x,y syndrome-Manipulation focus
15	25	8	3

*We had total of 51 surveys

Conclusions and Implications

As it was shown on the tables of results, majority of the students have multiple incomplete ways of thinking about derivative. They also have difficulties in using their knowledge about derivative to solve applied problems. If we can show that lack of complete ways of thinking about derivative impact students' abilities to solve applied problems, we can use that in enhancing our curriculum to ensure students' ways of thinking about derivative can be developed properly so they can solve the real world problems more effectively.

Future Plans

I am running the same survey this semester at three different differential calculus classes and then I am planning to divide the students based on their responses into three categories. These groups will be invited to participate in a task based interview. I am adopting a new approach for my interview similar to the method Selden, Selden, Hauk, & Mason (2000) and Selden, Mason & Selden (1994) used in collecting their written surveys. I am interested into investigating students multiple ways of thinking about derivative and how that affects their applied derivative problem solving. It seems from the preliminary data that students with multiple ways of thinking about derivative should be able to do better on the applied questions.

Questions for Discussion:

- The next phase of this project is to examine college mathematics instructors' knowledge of student thinking about derivative and application of derivative. What questions might be asked of these instructors to tap into their knowledge of the student thinking, including their knowledge of the impact of these differences on students' performance on tasks?
- What other interview protocols do you think can help us in investigating the correlations of students' ways of thinking and their methods in solving applied problems?
- Do you think these categories capture all the important aspects of thinking about derivative?

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