UTILIZING TYPES OF MATHEMATICAL ACTIVITIES TO FACILITATE CHARACTERIZING STUDENT UNDERSTANDING OF SPAN AND LINEAR INDEPENDENCE

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The purpose of this study is to investigate students' concept images of span and linear (in)dependence and to utilize the mathematical activities of defining, example generating, problem solving, proving, and relating to provide insight into these concept images. The data under consideration are portions of individual interviews with linear algebra students. Grounded analysis revealed a wide range of student conceptions of span and/or linear (in)dependence. The authors organized these conceptions into four categories: travel, geometric, vector algebraic, and matrix algebraic. To further illuminate participants' conceptions of span and linear (in)dependence, the authors developed a framework to classify the participants' engagement into five types of mathematical activity: defining, proving, relating, example generating, and problem solving. This framework could prove useful in providing finer-grained analyses of students' conceptions and the potential value and/or limitations of such conceptions in certain contexts.

Keywords*:* Span, Linear Independence, Linear Algebra, Mathematical Activity, Concept Image

The purpose of the study is to investigate student thinking about the important ideas of span and linear independence in linear algebra and to contribute to the body of knowledge regarding how individuals understand undergraduate mathematics. In particular, our research goals are:

1. To classify students' conceptions of span and linear (in)dependence.
2. To investigate how students use these conceptions to reason about relationships between span and linear (in)dependence.

The present study focused on interview data that elicited student reasoning about span and linear (in)dependence. We began our analysis through a grounded theory approach in order to identify student conceptions of span and linear (in)dependence. We noticed that, in coding students' concept images, our analysis was facilitated by noting the type of mathematical activity in which the students were engaged as they were sharing their ways of reasoning. In other words, the interview question to which they were responding had the potential of eliciting different aspects of the students' concept images. This is consistent with Vinner's (1991) notion of *evoked concept image.* For example, students' reasons why a claim was true or false revealed ways of thinking about the concepts involved different than did their response to "how do you personally think about this concept?" As such, we identified within the data set five mathematical activities in which students engaged during the interviews: describing, proving, relating, example generating, and problem solving. Within this study we show how these mathematical activities can be used as a lens to further refine characterizations of students' understanding of span and linear (in)dependence.

Given this framework, our refined research objectives are (a) to investigate students' concept images of span, linear (in)dependence, and relationships between the two concepts and (b) to utilize the mathematical activities of defining, example generating, problem solving, proving, and relating to provide insight into these concept images. Our results section briefly details the four concept image categories that grew out of our data: travel, geometric, vector algebraic, and matrix algebraic. We also define the five mathematical activities and provide an example of how coordination of the frameworks informed analysis of student thinking. Additional results will be discussed during the presentation.

Literature Review

There exists a growing body of research into student understanding of span and linear (in)dependence. For instance, Bogomolny (2007) makes use of APOS Theory (Dubinsky & McDonald, 2001) to examine how example generation tasks can influence student understanding of linear (in)dependence. Stewart and Thomas (2010) combine APOS with Tall's (2004) Three Worlds of Mathematics to generalize student understanding of linear independence, span, and basis according to the authors' genetic decomposition of the concepts. Hillel (2000) offers three modes of reasoning (geometric, algebraic, and abstract) within linear algebra, and Sierpinska (2000) suggests three modes of thinking in linear algebra: synthetic-geometric, analytic-algebraic, and analytic-structural. These modes of thinking are attributed to the historical development of linear algebra and align somewhat with Tall's three worlds framework in that the first focuses on spatial reasoning, the second on algebraic manipulation and representation, and the third on formal, theorem-based and axiomatic thinking.

While these studies expand our knowledge of student conceptions of span and linear (in)dependence, the current study differs in that our analysis of student conceptions are grounded with no a priori categorizations. Furthermore, our framing of student conceptions via the construct of mathematical activity adds a level of nuance into both powerful and problematic ways in which students reason about span, linear (in)dependence, and how they are related.

Setting and Participants

The data for this study comes from a semester-long classroom teaching experiment (Cobb, 2000) conducted in an introductory linear algebra course at a large public university. Classroom instruction was guided by the instructional design theory of Realistic Mathematics Education (RME) (Freudenthal, 1991), with the goal of creating a linear algebra course that builds on student concepts and reasoning as the starting point from which more complex and formal reasoning develops. The class engaged in various RME-inspired instructional sequences focused on developing a deep understanding of key concepts such as span and linear independence (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, in press), linear transformations (Wawro, Larson, Zandieh, & Rasmussen, 2012), Eigen theory, and change of basis.

The five students analyzed in this research - Abraham (a junior statistics major), Giovanni (a senior business major), Justin (a sophomore mathematics major), Aziz (a junior chemical physics major), and Kaemon (a senior computer engineering major) - participated in semi-structured individual interviews (Bernard, 1988) the week after final exams. Each interview lasted approximately ninety minutes. The purpose of the interview was to investigate how students reasoned about the concept statements that comprise the Invertible Matrix Theorem; the entirety of the interview protocol can be found in Wawro (2011). The current study considered only a portion of this data: students' conceptions of span, linear (in)dependence, and how they relate to each other. The interview questions analyzed in this study are given in Figure 1. Researchers analyzed video recordings and transcripts of the interviews, as well as all written work.

Methods

Videos and transcriptions of the participants' responses to Question 1a and 1b were iteratively analyzed. In the first analysis, the researchers focused on the logical progression of the participants' argumentation and what mathematical objects the participants attributed as 'acting' in different parts of their discussion (i.e. "the matrix spans $\mathbb{R}^3$" or "the vector moves in this direction"). A summarizing process that described the participants' general progression followed this analysis. A second analysis parsed out students' conceptions of linear (in)dependence and span, separating general discussion from instances when the interviewer directly asked the student to define the term. Quotes were drawn from the transcript and grouped by which concept the student was arguing with or describing. It was in this iteration of analysis that distinctions between types of activity became clear and led to the formation of the five categories of activity. In the next iteration of analysis the researchers categorized student quotes according to these five activities and separated the quotes according to span, linear independence, or linear dependence. These quote collections were then compared for categorical similarities and differences. At every stage of this process, the two researchers continually questioned and challenged each other's decisions, such as motivation for choice of categorization or interpretation of a student's quote.

"Suppose you have a 3 x 3 matrix *A*, and you know that the columns of *A* span R^3. Decide if the following statements are true or false, and explain your answer:"

Question 1a

The column vectors of *A* are linearly <u>de</u>pendent.

Follow-ups. Skip if redundant:

- "How do you think about span?"
- "How do you think about what it means for vectors to be linearly <u>de</u>pendent?
- "How does linear <u>de</u>pendence relate to span of a set of vectors?

Question 1b

The row-reduced echelon form of A has three pivots.

Follow-ups. Skip if redundant:

- "How do you think about what a pivot is?
- "How do pivots of a matrix relate to span of a set of vectors?"

Figure 1. Interview questions analyzed for this study.

Results

The participants in this study used a variety of language to describe their understanding of span, linear (in)dependence, and how the two concepts relate to each other. We organized this variety into four concept image categories: travel, geometric, vector algebraic, and matrix algebraic. We also identified five mathematical activities in which students engaged during the interview: defining, proving, relating, example generating, and problem solving. We present a table that coordinates these analyses for linear (in)dependence and briefly explain.

Categories of student conceptions

The *travel* category captures students' description of span and linear (in)dependence in terms indicative of purposeful movement. While this category is consistent with spatial and geometric reasoning, it is more specific in that it captures notions of "getting" or "moving" to locations in the vector space under consideration. The participants' travel conceptions of span were indicated by phrases such as "everywhere you can get" (stated by Justin when describing the span of a set of vectors) or "the vectors can take you anywhere [in $\mathbb{R}^3$]" (stated by Giovanni when describing what it means for vectors to span $\mathbb{R}^3$). With respect to linear independence, participants' travel conceptions included phrases like "[the vectors] only move farther away" (Justin) and "the vectors go in different directions" (Giovanni). A travel conception of linear dependence was generally indicated by phrases such as "then that would make that linearly dependent because I can, I can kind of get there and take that vector back" (Abraham), and "you can move 1 way on 1 vector, 2nd way, and then take the 3rd one back to the origin." (Aziz). These were often given as the inverse of phrases used to describe sets of linearly independent vectors. The *geometric* category includes student discussions in which a matrix or set of vectors are described as "covering an area," and when vectors are represented graphically on 2- or 3-dimensional axes. Most geometric examples of linear dependence showed either two collinear vectors or three vectors placed head to tail to form a triangle with one vertex at the origin.

The *vector algebraic* category captures participants' use of operations on algebraic representations of vectors in order to describe span and linear (in)dependence. This includes scalar multiplication, vector addition, and linear combination of vectors written as n x 1 matrices, or designated by variables (i.e. $2v + 3w$) as well as the use of the equation $A\mathbf{x} = \mathbf{b}$. Vector algebraic conceptions of span included "every vector you can make with linear combinations of the columns" (Justin) and "in order to span $\mathbb{R}^3$, vectors have to be different" (Giovanni). Vector algebraic conceptions of linear independence consisted of some form of the notion that only the trivial linear combination of linearly independent vectors would equal the zero vector. One participant, Abraham, described linear independence as when the equation $A\mathbf{x} = \mathbf{b}$ has one unique solution. This notion is included in this category since Abraham tended to focus on the product as a linear combination of column vectors of A rather than on the matrix as an entity. Vector algebraic conceptions of linear dependence include when a nontrivial linear combination yields the zero vector and the process of scaling one vector or taking a linear combination of two vectors in order to produce a vector that is linearly dependent.

The *matrix algebraic* category is based on instances in which participants used matrix-oriented algorithms such as Gaussian elimination through elementary row operations. Participants also demonstrated attention to whether a matrix was square and noted that elementary row operations maintain solution sets. Although these latter data describe matrices more generally and so interact with or inform conceptions of span, but are not conceptions of span, LI, or LD proper, they were also classified as matrix algebraic. The most prevalent notion of a matrix algebra conception was participants' reliance on the row-reduced echelon form of a matrix equaling the identity matrix (see Figure 2 for an example), with most of these participants discussing the pivots of the matrix. The participants frequently used this idea when discussing both span and linear independence – [square] matrices with column vectors that span the vector space or that are linearly independent row-reduce to the identity matrix. For a matrix algebraic conception of linear dependence, such a matrix would not row-reduce to the identity matrix.

Interviewer:	Um, how do you think about linear dependence in general?
Kaemon:	Um, dependence for me is just, first I just try to look with, if there's a matrix to see if it's, like, if it's already reduced, then I see if there's a variable, or I see that they're multiples, that they're 0 vector, just something to show it's dependent. And then if I can't find that, like, right away, then maybe I'll try to then, I don't know, try to reduce it, whatever, just until I could figure it out.
Interviewer:	So when you say, 'if I can look at it and see if it's dependent,' … do you have an idea behind what it means to be dependent?
Kaemon:	If I could find a linear combination between the vectors, then that doesn't, like, well zero's a scalar I guess, then, well no, that's dependent. It's, it's—every question always has like a different, like, indicator that it's dependent. So, like, it really depends what the question is. But usually, yeah, I just look, just look at the matrix first and then try to manipulate it, if it's not obvious.

Figure 2. Kaemon's matrix algebraic conception of linear dependence.

Types of Mathematical Activity

The construct of types of mathematical activities emerged from our grounded analysis of the data. As we analyzed students' understanding of span and linear (in)dependence in light of the concept image construct, we found ourselves continually drawn to notice the *type of activity* in which students were engaged as they responded to the interview questions. For instance, if a student spoke of span using a phrase such as "get everywhere," was that student engaged in explaining how span related to linear independence, explaining how s/he thought about the concept of span itself, or some other activity? As such, we identified five mathematical activities within the interview data: defining, proving, example generating, problem solving, and relating. We contend that considering the five mathematical activities provides insight into a student's understanding of a concept. We can think of these as facets of a student's interaction with the world based on what s/he understands a concept to be. These activities do not always occur in isolation. Furthermore, an activity may arise naturally based on the interview prompt or may occur spontaneously. Here we provide short descriptions of each activity; further detail and examples from the data set will be discussed during the presentation.

We use the term *defining* to mean the act of describing a concept's essential qualities. During the interviews, students were not asked to create definitions for concepts that were new to them, but rather to explain their notion of a concept's definition. As such, this use of defining may be of a slightly different connotation than the discipline-specific practice of defining (e.g., Zandieh & Rasmussen, 2010). Also, if students spontaneously (i.e., without prompting) described a concept, we put that within the "defining" activity. We use the term *proving* to mean the act of providing a justification to a claim. This reasoning process may be of various levels of mathematical rigor, and it may be carried out for the participant's personal conviction or to convince the interviewer. As such, we use "proving" similarly to Harel and Sowder's (1998) use of "the process of proving," which included the subprocesses of ascertaining and persuading. We use the term *relating* to denote any participant activity that compares, contrasts, or explains relationships between two concepts. The activities of proving and relating are similar, but distinct. We distinguish between these activities based on the participants' intentions. For instance, participants carried out relating activities while engaged in proof activity. Our analyses focused on the participants' purposes for expressing the relationship, and we categorized the activities accordingly. For instance, in Figure 2 Kaemon relates his notion of linear dependence to a matrix's appearance or operations that he might carry out on a matrix. The activity of *example generating* denotes when participants create cases of certain concepts or properties (e.g., a set of three linearly dependent vectors in $\mathbb{R}^3$). As with the other activities, this may be prompted by the interview explicitly or spontaneously done by the interviewee. Finally, the activity of *problem solving* is engaging in some calculation or reasoning with a specific goal to determine a previously unknown result.

Coordinated Analysis: Linear (In)dependence

A summary of the coordinated analysis between students' concept image categories and types of mathematical activity, within linear independence and dependence, is given in Table 1. Students' names are italicized to differentiate linear dependence from linear independence. It is worth noting that a student may understand linear (in)dependence in a way that is not indicated in this table; it may merely be the case that this particular interview did not evoke that understanding from the student at that time.

To lend insight into how a coordinated analysis informed the researchers, consider one student's struggle to coordinate his understanding of linear independence and span. Aziz's name appears in Table 1 in the travel row under the relating and example generating columns as well as in the geometric row under these same columns (among other places). These specific categorizations emerged as Aziz related linear dependence and span by generating geometric examples of linearly dependent vectors via a travel conception. Aziz generated these vectors, stating “they're linearly dependent, because you can use a combination of all 3 to get back to the origin” (geometric and travel). When trying to relate linear independence to span, however, Aziz stated that, “they're linearly dependent. Um...that's a problem I always thought, because if it’s…they move in 3 different directions, they should technically span $\mathbb{R}^3$.” Here, Aziz is referring to his previous statement that vectors spanning $\mathbb{R}^3$ need to move in three different directions (travel conception of span). Aziz makes sense of this seeming contradiction by noticing that the three vectors, “move on the same plane in 3 different directions, but not out of that plane.” This geometric conception of linear dependence allows Aziz to distinguish between the “3 different directions” of vectors that span $\mathbb{R}^3$ (3 dimensions) and vectors that are linearly dependent (3 ordinal directions in the same plane) and hence, make a meaningful comparison between span and linear independence when he states, “but it technically spans, no, makes it a plane in $\mathbb{R}^2$, $\mathbb{R}^3$. I got it, I figured it out.” Attending to the different activities that Aziz engaged in allows a more nuanced analysis of his different conceptions of span and linear dependence.

Table 1. Coordinated analysis of conception categories and mathematical activities.

Linear Independence/*Linear Dependence*					
	Defining	Relating	Proving	Ex. Generating	Pr. Solving
Travel	Aziz	Aziz/*Aziz*	Aziz	Aziz/*Aziz*	
	Abraham		*Abraham*		*Abraham*
	Giovanni/*Giovanni*	Giovanni/*Giovanni*			
	Justin/*Justin*	Justin/*Justin*			
Matrix Alg.	Aziz				
	Abraham	*Abraham*			*Abraham*
				Giovanni	
		Justin	*Justin*		
	Kaemon/*Kaemon*	*Kaemon*			*Kaemon*
Vector Alg.		*Aziz*	Aziz		
	Abraham	Abraham		*Abraham*	*Abraham*
				Giovanni	
				Justin	
	Kaemon			*Kaemon*	
Geometry		Aziz		*Aziz*	
				Abraham	
				Justin/*Justin*	

Conclusion

This proposal summarizes our work in categorizing students’ concept images of span and linear (in)dependence and our use of the construct of mathematical activity to provide insight into these conceptions. We note that the concept image categories that arose may be an artifact of the type of instruction and curriculum that these students experienced. We also note that the types of mathematical activity are not meant to be exhaustive; rather, these five activities were determined from analysis of this small data set. Analysis of classroom data or problem-solving interviews, for instance, would likely give rise to additional types of mathematical activity. As such, our future work involves a further examination and refinement of the framework of mathematical activity as a way to gain insight into students' conceptions of mathematical ideas. In addition, we also plan to examine additional data (classroom, a mid-semester interview) of these same five students in order to gain a more complete analysis of their understanding.

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