COOPERATIVE LEARNING AND TRAVERSING THE CONTINUUM OF PROOF EXPERTISE: PRELIMINARY RESULTS

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This paper describes preliminary results of a study aimed at examining the effects of working in cooperative groups on acquisition and development of proof skills. Particular attention will be paid to the varying tendencies of students to switch proof methods (direct, induction, contradiction, etc) based on their level of proof expertise. Namely, as students progress from novice to expert provers, they tend to change proof methods more frequently until they reach the final stages of development (Hart 1994).

Key words: Transition to Proof, Proof Writing Expertise, Proof Methods

Introduction

Although proof is essential to studying mathematics, much research in the past two decades shows that students struggle with constructing and validating proofs (Almeida, 2000; Harel & Sowder, 1998; Levine & Shanfelder, 2000, Moore, 1994; Selden & Selden, 2003a, 2003b; Weber, 2001; Weber, 2003). Several innovative course structures have been introduced for so-called bridge courses (Almeida, 2003; Bakó, 2002; Grassl & Mingus, 2004), but little dedicated research has been done on the effectiveness of such courses. However some common themes have emerged about the necessity for and efficacy of active learning strategies, and there is a general trend away from lecture and toward more student-centered models models. In particular, this can be seen within the Modified Moore Method community (McLoughlin, 2010).

Cooperative learning (CL) is one such model. "CL may be defined as a structured, systematic instructional strategy in which small groups work together to produce a common product" (Cooper, 1990). There are five specific features that, when combined, distinguish CL from other active and collaborative learning strategies: positive interdependence, individual accountability, student interaction, attention to social skills, and teacher as facilitator. While the efficacy of CL has been researched (Johnson & Johnson, 1991), the majority of this research has been undertaken with precollegiate populations.

Studies done on CL and active learning in the context of physics instruction (Deslauriers, et al, 2011; Heller & Hollabaugh, 1992; Heller, et al., 1992) give hope that CL could be effective in helping students acquire and develop their proof skills. This paper looks at some of the preliminary results of a study exploring the relationship between CL and proof-skill development. Specifically, the study was designed to examine how working in a CL seminar environment affected 1) students' attitudes about proof, 2) students' ability to construct proofs, and 3) students' abilities to validate student-generated arguments. The second of these will be addressed in this paper.

Hart (1994) compared expert and novice proof writers through use of a proof test. He categorized 29 undergraduate math majors by their level of proving expertise using three specific tasks from the test, and rated the students in one of four levels: Level 0: pre-understanding, Level 1: syntactic understanding, Level 2: concrete semantic understanding, Level 3: abstract semantic understanding (p. 56). He then examined the students' individual proof production processes noting similarities that arose among students at the same level of understanding. The mathematical context of Hart's study, abstract algebra, differed vastly from that of the study addressed in this paper, but some generally applicable findings were reported.

He noted that it is important not to try to get novice provers to perform like expert provers all at once; there is a continuum of expertise that must be traversed, even though the progression across it is often not smooth. In particular, he noted that expert provers switch proof methods less often than the most novice provers, but that the tendency to change plans increased between levels at all but the final step (pp. 59-60). This study examined whether a cooperative learning environment would enable students to become more expert provers and whether this would be marked by a similar change in tendency to change plans.

Methods

The subjects for this study were seven "seminar" students (five male, two female) and three "comparison" students (two male, one female). All students were undergraduates at a large, public university with a declared major or minor in mathematics.

All subjects took pre- and post- assessments of their proof construction skills via three proof prompts in basic number theory. The assessments of the seminar students were conducted in the presence of the researcher, and those subjects were asked to think aloud as they attempted to construct the proofs. The assessments of the comparison students were conducted in a group setting with each subject working independently and silently. At least 11 weeks passed between pre- and post- assessments for all subjects. In this paper, I will focus on the subjects' proof construction performance of all but one student. This particular student, Zach, did not make much effort on post-assessment, instead spending much of the interview complaining about the research methods. As a result, he performed more poorly on the post-assessment than he had on the pre-assessment (see Table 2).

The three prompts listed below were presented as true theorems dealing with elementary number theory concepts accessible to all of the subjects regardless of prior background and testing varying proof skills the researcher believes occur across content areas (see Table 1).

Assessment Item	Hypothetical Skill(s) Tested
1. Prove: If m^2 is odd, then m is odd.	• Use of indirect proof methods. • Avoidance of a more accessible converse argument.
2. Prove: If n is a natural number, then n^3-n is divisible by 6.	• Ability to identify pertinent subclaims and construct subarguments (divisibility by 2 and 3).
3. A *triangular number* is defined as a natural number that can be written as the sum of consecutive integers, starting with 1. Prove: A number, n, is triangular if and only if $8n+1$ is a perfect square. (You may use the fact that $1+2+...+k=\frac{k(k+1)}{2}$.)	• Use of the specifics of a definition to form a basis for a proof. • Ability to identify the logical implications of "if and only if" statements. • Use of previously established results (to prove $8n+1$ a perfect square implies that n is triangular, the result of item one needs to be applied).

Table 1. Assessment Items

Between assessments, the seminar students met with the researcher for eight, 90-minute sessions during which they worked on problem sets in cooperative groups. The cooperative groups were consistent throughout the study and were formed to be heterogeneous based on gender and on skill level as demonstrated on the pre-assessment. The members of each group spent a few minutes at the beginning of each session getting to know each other and 5-10 minutes at the end of each session doing group processing exercises. Both of these exercises facilitated the development of the social skills necessary for effective cooperative work, and the rotating roles (manager, explainer, skeptic, presenter) the students assumed each session assured their personal accountability and positive interdependence. After a brief introduction each session, the students worked with each other and the researcher functioned solely as a facilitator, encouraging the student-to-student interactions.

The problem sets dealt with function concepts, primarily injectivity and surjectivity, and the seminar group did not work with number theoretical concepts. This was done so that any changes from pre- to post-assessment would reflect changes in the subjects' proving skills independent of mathematical context.

The video recordings of the seminar students' assessments and of all seminar sessions were transcribed, and the transcriptions of the assessments were coded for instances in which subjects changed proof methods (direct, contradiction, contrapositive, induction) or switched to a different proof but returned later. All written proof attempts were also analyzed for correctness (0 – no progress or completely flawed, 1 – minimal progress or progress with substantial flaws, 2 – some progress with some flaws, 3 – substantial progress but incomplete or with minor flaws, 4 – correct proof). Specific errors appearing in the proofs were also coded according the list of common errors and misconceptions by Selden and Selden (2003b).

Preliminary Findings

Six of the seminar students showed dramatic improvement from pre-assessments to post-assessment. Those six were all able to reprove the results proved on the pre-assessment, though sometimes in a different manner, and all six were able to prove additional items. Despite the fact that their performances on the pre-assessment did not differ greatly from those of the seminar students, the three comparison students, all of whom were enrolled in at least one proof-based course, showed no noticeable improvement on the post-assessment.

Most of the six seminar students under consideration changed proof methods more frequently on the post-assessment than they had on the pre-assessment. The students who had the greatest change were those who had the weakest performances on the pre-assessment. There was only one student, Bill, who changed proof methods less frequently on the post-assessment than on the pre assessment, and he was one of the strongest students on the pre-assessment (see Table 2). These results mirror Hart's (1994) findings that as students progress from novice to expert provers, they are more likely to change plans mid-proof, except at the final stage of development when that tendency decreases. This progression was even shown on individuals' performances on specific tasks (see Table 3) illustrating the "rather unstable, irregular, developmental process" (Hart, 1994, p. 61).

Student	Pre-Assessment		Post-Assessment	
	Total Score	Number of Switches	Total Score	Number of Switches
Omar	0	0	4	1
Ursula	0	3	8	6
Ingrid	1	0	8	6
Ivan	4	3	6	3
Zach*	4	4	2	0
Nathan	6	1	8	2
Bill	6	5	9	4

Table 2. Seminar Student Performance on Assessments

The considered students who performed the best on the pre-assessment, Ivan, Nathan, and Bill, all had items on the post-assessment for which they changed plans less but performed as well or better. However, the lowest-performing students on the pre-assessment, Omar, Ursula, and Ingrid, all changed plans at least as many times on every item on the post-assessment as they had on the pre-assessment (see Table 3 for examples).

BILL	Description of Performance	Score	Number of Switches
Item 1 - pre	Produced a valid proof by contradiction.	4	3
Item 1 - post	Produced a valid proof by contrapositive	4	1
Item 2 - pre	Produced a proof that n^3-n is even, and recognized he was missing that 3| n^3-n.	2	3
Item 2 - post	Produced a proof that n^3-n is even, and recognized he was missing that 3| n^3-n.	2	1
Item 3 - pre	Manipulated the equation $8n+1=x^2$, but the manipulations were unproductive.	0	0
Item 3 - post	Produced a proof of both directions, but was missing the justification that 8n+1 is necessarily odd, so if it is a perfect square, then it is the square of an odd number.	3	2
URSULA			
Item 1 - pre	Produced an empirical contradiction argument with use of a single example, m=2.	0	1
Item 1 - post	Produced a valid proof by contrapositive.	4	2
Item 2 - pre	Did not identify subgoals, attempted a proof by induction, but could not get to conclusion even though the work was error-free.	0	1
Item 2 - post	Identified subgoals, attempted a proof by contradiction again and successfully proved that 3| n^3-n. Had the work to get 2| n^3-n, but did not recognize that k^2-k is necessarily even.	2	3
Item 3 - pre	Correctly stated givens and goals, attempted to use $1+2+...+k=\frac{k(k+1)}{2}$, but set up $8(\frac{k(k+1)}{2})+1=triangular$. Made no progress from there.	0	1
Item 3 - post	Proved that n triangular implies 8n+1 is a perfect square. Made no progress on reverse direction.	2	1

Table 3. Student Performance on Individual Items

Based on Hart's (1994), these data show that not only were the seminar students able to improve their performances between assessments, but that they matured along the spectrum of novice to expert provers. Additionally, the stark difference seen between the improvement of the students in the seminar group and the lack of improvement of those in the comparison group indicates that the cooperative seminar was key to the seminar students' improvement. This study suggests the need for further study along these lines in randomized comparative trials of the effects of cooperative learning on students' development of proof skills. Also, while proof researchers often talk about proving skills as if they were not context-dependent, this study shows strong support that some proving abilities are truly context-independent, but this needs to be studied in much more detail.

Questions

1. What other indicators that the research subjects progressed from more novice to more expert provers might I look for?
2. The tasks Hart used to define expertise levels were very different from my tasks. Is it reasonable to draw similarities and parallels from his work given that I can't apply the levels to my own students directly?
3. What proof skills do you think may be context-independent? Do you think there are any?

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