ON MATHEMATICS MAJORS' SUCCESS AND FAILURE AT TRANSFORMING INFORMAL ARGUMENTS INTO FORMAL PROOFS

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In this paper, we examine 26 instances in which mathematics majors attempted to write a proof based on an informal explanation. In each of these instances, we represent students' informal explanations using Toulmin's (1958) scheme, we use Stylianides' (2007) conception of proof to identify what one would need to accomplish to transform the informal explanation into a proof. We then compare this to the actions that the participant took in attempting to make this transformation. The results of our study are categories of actions that led students to successfully construct valid proofs and actions that may have hindered proof construction.

Key words: Proof; Proof construction; Informal argument

Introduction and Research Questions

Proving is central to the practice of mathematics. Consequently, a goal of most upper-level mathematics classes is to improve mathematics majors' abilities to construct proofs. Unfortunately, numerous studies have documented mathematics majors' difficulties with writing proofs (e.g., Alcock & Weber, 2010; Hart, 1994; Moore, 1994). Research in this area has identified particular difficulties that students have with proof writing, such as a limited understanding of the mathematical concepts being studied (Hart, 1994) and not knowing how to begin when asked to write a proof (Moore, 1994). However, exactly how undergraduates can and should write proofs remain important questions in undergraduate mathematics education.

The proofs that undergraduates are asked to write in their advanced mathematics courses are required to be formal. That is, these proofs are expected to begin with definitions, axioms, and/or appropriate assumptions and proceed deductively to reach a desired conclusion, often while employing logical syntax. However, although a proof that is produced is required to be formal, the process of producing this proof may be far less rigorous. Numerous mathematics educators advocate that mathematics majors should base at least some of their proofs on informal arguments (e.g., Garuti et al, 1996; Raman, 2003; Weber & Alcock, 2004). For instance, although it is not valid to infer a property about a concept by the inspection of a single example or a diagram of the concept, the insights gained from studying a diagram or example can suggest properties that may be true and useful for constructing a valid proof.

In recent years, the literature on mathematics education has moved beyond simply recommending that students base proofs off of informal explanations and has begun to analyze the *types* of informal arguments that students can and cannot formalize into formal proofs (e.g., Alcock & Weber, 2010; Pedemonte, 2007; Pedemonte & Reid, 2011). The research questions in this proposal focus on the cognitive actions an individual can take to formalize an argument. In particular, we investigate the following:
(i) When mathematics majors give an informal explanation for why an assertion is true but are unable to prove this assertion, why were they unable to do so?
(ii) When mathematics majors successfully transform informal explanation into a valid proof, what actions did they take that enabled them to do so?

Related Literature

Garuti et al (1996) introduced the construct of *cognitive unity* to describe a continuity between the informal reasons that students have for believing a mathematical assertion is true and the proof that they produce of that assertion, arguing that it is desirable that these two be linked in students' proof productions. This is consistent with Weber and Alcock (2004) citing the benefits of semantic proof productions (i.e., proofs based upon informal representations of concepts) and Raman's (2003) call for students to produce proofs based upon a key idea, meaning that students should translate personal intuitive arguments into public formal proofs.

There are two empirical findings that support these recommendations. The first is that these are consistent with the practice of many mathematicians, who also base their formal proofs on informal explanations (e.g., Burton, 2004; Raman, 2003). Second, case studies reveal that *some* students who engage in proof writing in this way can be highly successful (e.g., Alcock & Weber, 2010; Gibson, 1998). However, there are also two reasons to question the viability of these findings. First, as Duval (2007) emphasizes, the structure of informal explanation and proofs differs greatly, notably with respect to the epistemological status of assertions within these arguments. Others (e.g., Alcock, 2010) argue the transition between an informal argument and a formal proof is cognitively difficult. Second, there are also many instances in the literature of students not being able to prove a statement despite seeing why an assertion is true (e.g., Alcock & Weber, 2010; Pedemonte, 2007). This emphasizes the need for a greater understanding of the process of basing a formal proof off of an informal argument.

Recently, Pedemonte (2007) and Pedemonte and Reid (2011) advanced the discussion of this issue by introducing the construct of the *structural distance* between an informal explanation and a formal proof, measuring how easy or difficult it would be to translate the former to the latter based on the type of warrants (in the sense of Toulmin, 1958) that were employed. Pedemonte (2007) argued that informal explanations based on process-based generalizations or abductive inferences had a shorter structural distance to proof than those based on results-based generalizations; consequently, students who generated the former types of explanations had more success at proof writing.

Theoretical Framework

In this study, we follow Pedemonte (2007) and Pedemonte and Reid (2011) in using Toulmin's (1958) framework to analyze students' informal explanations and formal proofs. Furthermore, we adapt Stylianides' (2007) characterization of proof as a normative framework to highlight the gap between explanation and proof and distinguish between valid and invalid proofs. We view an argument as consisting of a series of inferences. Applying Toulmin's framework to each inference, we say that an inference consists of a *claim* (the conclusion that is being advanced), *data* (the facts that support the claim), and a *warrant* (the reason that the claim is necessitated by the data). We note that as arguments are based on a series of inferences, the data or warrant for a new inference may have been the conclusion of a previous inference.

Stylianides (2007) argued that arguments should meet three standards to qualify as a proof; the argument should (i) use inference methods that are valid, (ii) be based upon facts that are true and acceptable, and (iii) use representations that are appropriate, both to the audience who is observing the proof and to the larger mathematical community. Within our Toulmin perspective, for an argument to be a proof, the warrants must be deductive, the initial assertions (i.e., the initial data) being used must be definitions and established facts, and the claims in the proof should be expressed in standard mathematical syntax. We note that informal explanations may not satisfy these standards. Starting points of mathematical argumentation need not be definitions, but may be other representations of mathematical concepts, such as a diagram.

Warrants may not be deductive, but may be perceptual (e.g., "f is increasing because it looks that way on the graph") or inductive (e.g., "the first four perfect numbers are even, so all perfect numbers are even"). Finally, in informal arguments, claims may be expressed informally in everyday language (e.g., "goes up" instead of "increasing").

To illustrate, consider a hypothetical student who was informally justifying why $4x^3 - x^4 = 30$ has no solutions. She might say, "the graph of $4x^3 - x^4$ seems to have a maximum value of 27. Therefore, it will never reach 30 and no solution can be obtained". In the first inference, the claim, "$4x^3 - x^4$ seems to have a maximum value at 27" is inferred from the data (the graph of the function $f(x)=4x^3 - x^4$) through a perceptual warrant (it "seems" that way from the graph). Transforming this argument into a proof would involve stating the claim more rigorously (e.g., "$4x^3 - x^4 \leq 27$ for all real x"), basing it on acceptable data (e.g., the symbolic representation of $f(x)$ rather than a graphical representation), and using a deductive rather than a perceptual warrant (e.g., using calculus to find global maxima of functions). In this study, we investigate what types of cognitive processes might help or hinder students to make these translations.

Methods

Participants. Twelve recent graduates who had majored in mathematics at a large public university in northeastern United States agreed to participate in this study. Students had all taken an introductory proof course at this university.

Materials. Participants were asked to construct seven proofs in calculus and seven proofs in linear algebra. These tasks were chosen so that they could be successfully completed either by syntactic or semantic reasoning, in the sense of Weber and Alcock (2004). That is, we believed it was plausible that students could construct a proof either by symbolic manipulation and logical deduction or by translating an informal explanation into a proof.

Procedure. Participants met individually with a member of our research team for two task-based interviews each lasting approximately 100 minutes. Participants were videotaped as they completed the proofs. They worked on each proof, one at a time, either until they produced what they believed was a proof, they felt they could make no further progress, or ten minutes had elapsed. At any time, participants were allowed to ask for a sheet containing the formal definition and an example object of any concept involved in the study. They were also given access to graphing software. After each proof attempt, participants were asked to describe their thought process as they worked on the proofs.

Analysis. We first flagged each of the 168 collective proof attempts (12 students each attempting 14 proofs) for instances of an informal argument. We defined an informal argument as an argument containing at least two inferences with at least one of the inferences being based on a warrant that was non-deductive. There were 26 such informal explanations. We analyzed each informal explanation and the corresponding proof (when a proof was written) using Toulmin's scheme. We coded each warrant as an instance of logical deduction, perceptual reasoning, results-based generalization, process-based generalization (cf. Harel, 2002), abductive inference (cf., Pedemonte, 2007), or an abductive warrant. An abductive warrant involved where participants conjectured a principle that might explain why data implied a conclusion (e.g., upon observing that $\sin x$ and $\sin^3 x$ were odd functions, guessing that the product of odd functions was odd). We then coded each informal explanation as being correct (i.e., each inference claim was true) or incorrect and each proof as being valid or invalid. Our qualitative analysis focused on the ways that participants failed or succeeded to transform their informal explanations into valid proofs by focusing on how they:

(i) transformed their initial data from unacceptable facts (e.g., unjustified assertions, informal representations such as graphs) to acceptable facts (e.g., formal definitions),
(ii) transformed arguments based on non-deductive warrants to deductive ones,
(iii) and expressed the assertions in the informal explanation more rigorously.

Results

In the presentation, we will present a list of categories for how students successfully and unsuccessfully attempted to transform informal explanations into proofs. To illustrate the type of analyses that we will discuss, we present one interesting case in detail here. P1 was attempting to prove that $x^3 + 5x = 3x^2 + \sin x$ only had a solution at $x = 0$. After several false starts, P1 graphs $f(x) = x^3 + 5x - 3x^2$ using the computer graphing software and says:

"Yeah... [the graph] doesn't have a bump so I guess it's going to go through that region really and then only going to be between zero and one in a really small area and I guess I just need to prove that it doesn't cross more than once in that area. Oh, and that in that area it's going to be strictly increasing and that sine is also going to be strictly increasing, and that it can only cross once."

Based on these and subsequent comments, we interpreted this argument as saying that a solution occurs when $f(x) = \sin x$. Since the range of sin x is [-1, 1] a solution can only occur for x values where $-1 \leq f(x) \leq 1$. She further notes that $f(x)$ is strictly increasing in this area (or "doesn't have a bump"). She concludes that the only solution to $f(x) = \sin x$ is $x = 0$, inferring that increasing functions can only intersect once. We coded the argument as follows:

Inference 1: Claim: $f(x)$ is between -1 and 1 for a small region.
Data: The graph of $f(x)$.
Warrant: Perceptual. $f(x)$ appears to be between -1 and 1 only briefly based on its graph.

Inference 2: Claim: A solution to $f(x)=\sin x$ can only occur when $f(x)$ is between -1 and 1.
Data: (inferred) The range of sine is [-1, 1].
Warrant: (inferred) Algebraic-deductive. If $f(x)>1$ and $g(x)\leq 1$, then $f(x)\neq g(x)$.

Inference 3: Claim: $f(x)$ is strictly increasing in the region (described in Inference 1)
Data: The graph of $f(x)$.
Warrant: Perceptual. The graph of $f(x)$ is increasing.

Inference 4: Claim: $f(x)$ and $\sin x$ only intersect once in that region (described in Inference 1)
Data: $f(x)$ is increasing (from Inference 3) and $\sin x$ is increasing in this region.
Warrant: (inferred) Abductive. Increasing functions can only intersect once in a region.

We coded the warrant in Inference 4 to be an abductive warrant, or in Pedemonte and Reid's words, a "creative warrant". That is, we believe that in trying to determine why $\sin x$ and $f(x)$ only intersected once in that region, P1 made the inference that increasing functions could only intersect once. This inference, and consequently the informal explanation in its entirety, is not correct (e.g., $f(x)=2x$ and $g(x)=2x + \sin x$ are increasing functions that intersect infinitely often). Transforming this argument into a proof would require (a) expressing the ideas contained in the claim more clearly (e.g., specifying precisely what the "region" in the argument referred to), (b) basing the argument on the algebraic rather than the graphical formulation of $f(x)$, (c) providing a deductive, rather than graphical, warrant to support Inference 1 and Inference 3, and (d) recognizing the warrant in Inference 4 was invalid and offering an alternative justification.

P1 accomplished (a), (b), and (c). She justified Inference 3, noting that $f'(x)=3x^2 - 6x + 5 = 3(x+1)^2 + 2$, which was strictly positive. This illustrates that some students are able to provide deductive backing for non-deductive warrants if they are aware of proving schema to establish the claims in question. (Note here the schema is one establishing $f(x)$ is increasing by showing $f'(x)>0$). She also expressed Inference 2 more rigorously. In her proof, after justifying why it was sufficient to show that *f(x)* = sin *x* only had one solution and demonstrating that *f(x)* was strictly increasing, she wrote, "Note f(-π/2)=-14.32<-1 and f(π/2)=4.276>1. Since -1≤sin *x*≤1, the solution can only have a real solution in the range –π/2<x<π/2". Note that P1 changed the range of her original argument which would have required computing f^{1}(-1) and f^{1}(1). This modification also obviated the need to justify Inference 1.

However, P1 did not reconsider the invalid abductive warrant for Inference 4. As this warrant was not in the established theory of calculus (and indeed was false), the resulting proof was not valid. We note that the inference of an abductive warrant, even an invalid one, need not cause a proof to be invalid. We have documented other cases in which participants assessed the plausibility of their inferred warrants with examples, observed that they were false, and successfully reformulated their argument. Hence, challenging one's abductive inferences and attempting to provide deductive backing for these inferences are important for proving success.

Summary and Significance

In this study, we have identified several reasons that participants are unable to transform informal explanations into valid proofs, including not investigating the veracity of one's abductive inferences (as discussed above) and simply changing the representation of the argument (i.e., using more formal language) without addressing the warrants by which the inferences in the argument were based (a fairly common occurrence that we did not discuss due to space limitations). We have also identified several ways that participants were able to base proofs on valid arguments, including identifying deductive schemas that could be used to justify inferences based on perceptual or results-based generalizations (as illustrated above). If we expect students to successfully use informal arguments as the basis for proving, as numerous authors suggest (e.g., Garuti et al, 1996; Raman, 2003; Vinner, 1991; Weber & Alcock, 2004), then it is incumbent upon the instructors to have students develop strategies such as these that allow them to do so.

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