CALCULUS STUDENTS' UNDERSTANDING OF VOLUME

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Researchers have documented difficulties that elementary school students have in understanding volume. Despite its importance in higher mathematics, we know little about college students' volume understanding volume. This study investigated calculus students' understanding of volume. Clinical interview transcripts and written responses to volume problems were analyzed. One finding is that some calculus students, when asked to find volume, find surface area instead and others blend volume and surface area ideas. We categorize students' formulae according to their volume and surface area elements. Clinical interviews were used to investigate why students might find surface area when asked for volume. We found that some students believe adding the areas of an object's faces measures three-dimensional space. Findings from interviews also revealed that understanding volume as an array of cubes is connected to successfully solving volume problems. This finding and others are compared to those for elementary school students. Implications for calculus teaching and learning are discussed.

KEYWORDS Student thinking, Calculus, Volume, Surface Area

1 INTRODUCTION

Many calculus topics involve volume: optimization and related rates in differential calculus, volumes of solids of revolution and work problems in integral calculus, and multiple integration, to name a few. Although volume shows up in these places, researchers have focused on elementary school students' difficulties with volume and little is known about older students' understanding of this idea. This study extends knowledge about students' understanding of volume and, in using non-calculus tasks, builds a foundation for studying volume understanding in calculus contexts.

We conducted this study within a cognitivist framework, giving students mathematical tasks and analyzing the reasoning underlying their answers. This is consistent with the cognitivist orientation toward focusing on "the cognitive events that subtend or cause behaviors (e.g., [a student's] conceptual understanding of the question)" (Byrnes, 2000, p.3). We collected written survey data and conducted clinical interviews to investigate the following research questions:

The research questions investigated are:
1. How successful are calculus students at volume computational problems?
2. Do calculus students find surface area when directed to find volume?
3. If calculus students find surface area when directed to find volume, what thinking leads them to do so?

Our major finding is that nearly all students correctly calculate the volume of a rectangular prism, but many students perform surface area calculations or calculations that combine volume and surface area elements when asked to find the volume of other shapes.

2 ELEM. SCHOOL STUDENTS' UNDERSTANDING OF VOLUME

There is a paucity of literature about calculus students' volume understanding, though it is known that some calculus concepts involving volume (e.g., related rates, optimization, and volumes of solids of revolution) are difficult for students (Martin, 2000; Tomilson, 2008; Orton, 1983). Research about elementary school students' volume understanding provided a basis from which the researcher investigated calculus students' volume understanding. Findings from research about elementary school students' volume understanding suggest that this population has trouble with arrays, formulae, and cross-sections.

Volume computations rely on the idea of an array of cubes, a representation with which elementary school students struggle (Battista & Clements, 1996; Curry & Outhred, 2006). Two difficulties students have are (1) understanding the unit structure of an array and (2) using an array for volume computation. Battista and Clements (1996) found that only 23% of third graders and 63% of fifth graders could determine the number of cubes in a 3x4x5 cube building made from interlocking centimeter cubes. One source of this difficulty is not seeing relationships between rows, columns, and layers, leading students to double-count innermost and edge cubes or viewing the array "strictly in terms of its faces" (Battista & Clements, 1998, p. 229). In other words, it seems that some elementary school students are thinking about surface area when asked about volume.

Other findings indicate that some elementary school students use area and volume formulae without understanding them (De Corte, Verschaffel, & van Collie, 1998; Fuys, Geddes, & Tischler, 1988; Nesher, 1992; Peled & Nesher, 1988). For example, Battista and Clements (1998) found that some students' strategies involved "explicitly using the formula *L x W x H* with no indication that they understand it in terms of layers" (Battista & Clements, 1998, p. 229).

Lastly, findings indicate that identifying the shape of a solid's cross-section is difficult for students (Davis, 1973). This finding is important because some volumes can be thought of as V=Bh where B is the base of the solid and the base is, in fact, a cross-section. Students having difficulty finding the shape of a cross-section would thus have difficulty using the V=Bh formula. This finding carries particular importance if it is also true for calculus students, as volumes of solids of revolution problems require identifying the shape of a cross-section.

3 RESEARCH DESIGN

The data analyzed were from written surveys completed by 198 differential calculus students and 20 clinical interviews with a subset of those students. Data collection had two phases: first, students completed the written tasks (modeled after those used in research with elementary school students) and data were analyzed based on those researchers' methods and a Grounded Theory inspired approach (Corbin & Strauss, 2008) where necessary. Clinical interviews (Hunting, 1997) were used to investigate patterns from the written data; that is, interview subjects were selected because their answers on written tasks represented an emergent category. This methodology allowed for a quantitative analysis of a large number of written responses and a qualitative analysis of student thinking about those written responses.

Written survey tasks consisted of diagrams of solids with dimensions labeled. Students were directed to compute the volume and explain their work. The rectangular prism task is shown below; the other tasks are included in Appendix A.

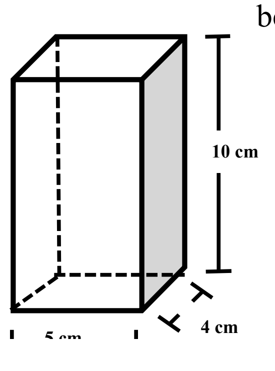


Figure 1. Volume of Rectangular Prism Task

Clinical interview tasks were the same as the written tasks. Interviewees were asked to re-work their solutions, thinking aloud as they did so. Questions were asked to probe student understanding, such as “Can you tell me about that formula? Why is the 2 there?” and “Can you tell me why that [formula] finds volume?” Interviews were audio recorded and transcribed.

Analysis of written surveys. Data were analyzed using both the methods of other researchers and an approach inspired by Grounded Theory (Corbin & Strauss, 2008). This entailed first looking for patterns in a portion of the data and using patterns to form categories, followed by forming category descriptions and criteria. Those criteria were then used to code all the data, testing the criteria until new categories ceased to emerge. The analysis resulted in three categories for students’ work: volume, surface area [instead of volume], and other. The categories and their descriptions are as shown in Table 1.

Found Volume	**Found Surface Area Instead of Volume**	**Other**
Magnitude is the correct magnitude of the object’s volume, or magnitude is incorrect for the object’s volume but the work/explanation is consistent with volume-finding (i.e., multiplication or appropriate addition)	Magnitude is the magnitude of the object’s surface area or the student work/explanation contains evidence of surface-area like computations, such as addition. To allow for computational errors, magnitude may or may not be the actual magnitude of surface area.	Student found neither volume nor surface area

Table 1. Categories of Students’ Responses

We used these categories to develop coding algorithms for the four volume computations. See Appendix B for a sample algorithm.

Analysis of clinical interviews. As interview data included both transcripts and students’ written work, there were two parts to the analysis. First, written data were categorized according to the aforementioned algorithms. Second, transcripts were used to investigate the thinking involved in answers for each category. This was useful because formulae themselves do not tell the whole story; for instance, some students explained the formula $2\pi r^2 h$ as a V=Bh formula, mistakenly believing that the area of a circle is $2\pi r^2$. Other students described $2\pi r^2$ as accounting for the area of two circles. The former is a volume idea; the latter a surface area one. This led us to categorize students’ formulae according to their area and volume elements, with the categorizations based on how students talked about those formulae. We begin with this in the Results section, then present the frequency of volume and surface-area finding for the four shapes.

4 RESULTS

We believe there is an important link between students’ formulae and their reasoning: that is, our data leads us to believe that students’ formula are not (as is commonly assumed) remembered or misremembered, but are instead representative of ideas students have about volume. This finding, based on the synthesis of interview data with written work, led us to categorize students’ formulae according to their surface area and volume elements. What we mean by "surface area and volume elements" is what we alluded to in discussing how the 2 in $2\pi r^2 h$ might be from an ill-remembered area formula and might be from accounting for two bases. Categorizing students' formula in this way gave us the categories and component formulae shown in Table 3. Note the appearance of $2\pi r^2 h$ in both the "incorrect volume, no surface area element" and "surface area and volume elements" categories, per the reasoning stated above.

Correct volume	Incorrect volume, no surface area element	Surface area and volume elements	Surface area	Perimeter
$\pi r^2 h$	$2\pi r^2 h$ $(1/3)\pi r^2 h$ $(1/2)\pi r^2 h$ $(4/3)\pi r^2 h$ πrh $(1/2)\pi rh$ $h*d*r$	$2\pi r^2 h$ $2\pi rh$ $2\pi r + \pi rh$ $\pi r^2 + 2\pi d$ $2\pi r^2 + 2rh$	$2\pi r^2 h + 2\pi rh$ $2\pi r^2 h + \pi dh$	$d+h$

Table 2. Categories for student responses to the cylinder task

This table includes all formulae that appeared in students' written work and interviews. Interview data provided help in placing the formula, and interview data are the basis of our claim that students' formulae are a reflection of their reasoning. For instance, consider Nell's reasoning about the volume of the cylinder:

> **Nell:** I don't know the formula for this one. Two pi r squared… times the height. Sure. We'll go with that one. So you have two circles at the ends, which is two pi r squared… you have two pi r squared because that's the area on the top and the bottom so you can just double it, then you have to times it by the height.
> **Interviewer:** Why do I have two areas?
> **Nell:** You have two circles.

The inclusion of the areas of the bases of a shape (what Nell calls the top and bottom) is part of finding surface area. However, Nell was not thinking about surface area, she was thinking about volume. This is evidenced by the following excerpt:

> **Interviewer:** What about this multiplying by the height? Why do we do that?
> **Nell:** It gives you the space between the two areas. Volume is all about the space something takes up so you need to know how tall it is.

Nell's reference to the space between two areas is indicative that she was thinking about volume. However, as previously stated, her formula ($2\pi r^2 h$) included a surface area idea. We thus put the formula $2\pi r^2 h$ in the "surface area and volume elements" category (see Table 3). It is also included in the "incorrect volume, no surface area elements" because other students talked about this formula as *area of base times height* where the area of the base was $2\pi r^2$. In this case, the two is not a nod to two bases, it is an incorrect formula for area but correct reasoning for volume.

Nell was not the only student who thought about including both circles when finding volume: Jo went back and forth about whether she should use the formula $2\pi r^2 h$ or $\pi r^2 h$. The interviewer asked her to make the case for both one and two circles as a way to investigate her reasoning:

> **Jo:** The area of the circle is pi r squared times the height, but I can't decide if I need one or two circles.
>
> **Interviewer:** Convince me that you need two circles.
>
> **Jo:** You need two because you have the top and the bottom of the cylinder. But you don't actually need two… you just need the one. Because you get the area of the circle and you multiply it by the height… the circle is the same throughout the whole layer so you just multiply it by the height.

Jo's final reasoning was correct, but it's noteworthy that her initial response to the problem involved a surface area idea. Thus, despite her correct final response, we believe this is evidence that some students have mixed and combined surface area and volume ideas. An additional interesting result is that the frequency of this phenomenon appears to be shape-dependent. This is evident in the percentages of students who fell into each category, shown in Table 3.

	Rectangular prism (n=198)	Cylinder (n=198)	Triangular Prism (n=122)	Trapezoid (n=7)
Found Vol.	194 (98.0%)	172 (86.9%)	95 (77.9%)	5 (71.4%)
Found SA	3 (1.52 %)	10 (5.1%)	17 (13.9%)	2 (28.6%)
Other	1 (0.5%)	16 (8.0%)	10 (8.2%)	0 (0%)

Table 2. Percentage by task

Analysis of written data indicated that some differential calculus students find surface area when directed to compute volume. However, the percentage of surface-area-finding students varies by shape, with few students (1.52%) finding surface area for the rectangular prism, 5.1% of students finding surface area for the cylinder, and 13.9% of students finding surface area for the triangular prism. A much higher percentage of students (71.4%) found surface area for the trapezoid; however, as n=7, this may not be a representative sample.

5 CONCLUSIONS AND IMPLICATIONS

Findings indicate that student success with computational volume problems differs by shape. Students were extremely successful with objects such as the rectangular prism, but struggled with the assumedly less-familiar trapezoidal prism. This has implications for volume-finding in calculus; for instance, volumes of solids of revolution are rarely elementary shapes. A second finding is that some calculus students find surface area instead of volume, either thinking that adding the areas of faces finds volume, or having formed an amalgam of surface area and volume ideas.

One implication for instruction is that instructors might use student-generated formulae to diagnose their ideas. Viewing students' formulae in terms of surface area and volume elements may provide clues to the ideas students hold about surface area and volume, and asking students about the formulae they use may provide insight as to the ideas they hold. An additional implication for instruction is to provide opportunities for calculus students to revisit and strengthen their understanding of surface area and volume, including unpacking the formulae for each. Further, using the ideas of the shapes of cross-sections and bases could be useful not only in understanding the volume of geometric solids for which V=Bh can be applied, but might aid students when they learn volumes of solids of revolution.

A suggestion for further research is to find out if students' volume/surface area difficulties interact with their learning of related rates, optimization, and volumes of solids of revolution. It would be interesting to know what happens for a student with an amalgam of surface area and volume tries to optimize the surface area for some given volume and whether difficulties with this are related to underlying issues with surface area/volume, issues with the calculus, or both.

Appendix A: Tasks

2b. What is the volume of the cylinder? Explain how you found it.

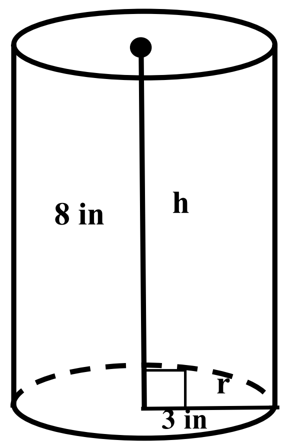


Figure 2. Cylinder Task

3. What is the volume of the object? Explain how you found it. <u>Note:</u> The figure is not drawn to scale.

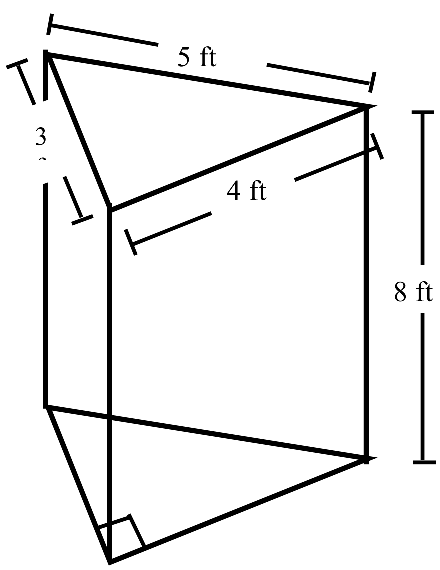
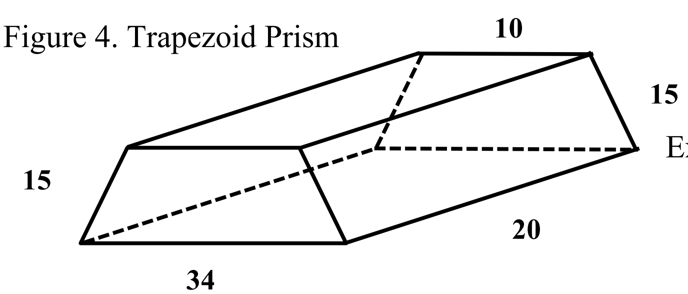


Figure 3. Right Triangular Prism Task

Figure 4. Trapezoid Prism



6.What is the volume of the prism? Explain how you found it.

Appendix B: Example Coding Algorithm

Cylinder coding algorithm:

Correct volume = $\pi r^2 h = \pi(3^2)(8) =$ *72π [units³]*

Correct surface area = $2\pi r^2 + 2\pi rh = 2\pi(3^2) + 2\pi(3)(8) =$ *66π [units²]*

1. Did the student write the formulae $\pi r^2 h$ or $2\pi r^2 h$? Did the student write 72π or 144π? If so, categorize as "found volume." If not, proceed to #2.
2. Did the student write πr^2 + ________ or $2\pi r^2$ + ____________where ________ is something that looks like it might be *πdh* or some other computation that looks like an area of a lateral face? Did the student write 66π? In either case, categorize as "found surface area instead of volume." If not, proceed to #3.
3. Categorize as "other."

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