On the Plus Side: A Cognitive Model of Summation Notation
(Contributed Research Report)

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This paper provides a framework for analyzing and explaining successes and failures when working with summation notation. Cognitively, the task of interpreting a given summation-notation expression differs significantly from the task of expressing a long-hand sum using summation notation. As such, we offer separate cognitive models that 1) outline the mental steps necessary to carry out each of these types of tasks and 2) provide a framework for explaining why certain types of errors are made.

Key words: *summation notation, calculus, real analysis, cognitive models, APOS*

Introduction

In the Spring of 2011, we began an Advanced Calculus teaching experiment with pairs of students. The purpose of this experiment was to lay the groundwork for a new instructional sequence for Advanced Calculus (Real Analysis). The first sequence of tasks had the students investigate notions of area, with increasing formality and rigor, in order to motivate the study of sequential limits. In the midst of this experiment, two of our students encountered interesting challenges in using summation notation to talk about area. At last year's RUME conference we detailed how those two students struggled, but the data was insufficient for us to warrant any strong claims about *why* they struggled (Strand, Zazkis, & Redmond, 2012).

The research reported here is a follow-up study designed to further understand the cognitive complexity of summation notation and contribute a framework that could help explain students' difficulties. In particular, we sought to answer the following questions:

Are student difficulties with summation notation caused *or* revealed *by working within the context of rectangular approximations to area under a curve?*

What are some plausible explanations for the difficulties students encounter?

Theoretical Perspective

Our cognitive models are similar in structure and purpose to the genetic decompositions of APOS theory. A *genetic decomposition* "is a model of cognition: that is, a description of specific mental constructions that a learner might make in order to develop her or his understanding of the concept" (Brown, DeVries, Dubinsky, & Thomas, 1997). The difference between our models and a genetic decomposition is that we are not describing how the effective use of summation notation is developed by a student but rather what kinds of mental constructions are necessary in order to use the notation effectively.

In general, these "mental constructions" can be broken down into *actions, processes, objects,* and *schema.* This breakdown is a framework that is often referred to by the acronym APOS (ibid). *Actions* are procedures or transformations that are performed on objects, often in a very step-by-step manner. Through interiorization, a sequence of actions can be reflected upon and envisioned and analyzed without needing to be carried out. When an individual interiorizes a sequence of actions we say that they have constructed a *process.* When an individual is able to reflect on a process as a whole and even apply other actions to that process, we say that the individual has encapsulated that process into an *object.* Actions, processes, and objects can be coordinated into a *schema*.

These models of cognition can be useful for instructional design as well as helping to explain the causes of specific student errors and non-standard conceptions. Below we outline our cognitive models, which describe how the effective use of summation notation involves the successful coordination of a number of actions and processes into a coherent schema. In our Results section we will demonstrate how these models may be used to provide plausible explanations for some observed survey responses.

Expanding a summation-Notation Expression: summation notation can be thought of as a schema, encoding a coordinated set of mental activities or procedures :

- *Iteration* - running through the iterating values of the index
- *Function Evaluation* - This could be a conventional function in the summand of the notation, but also captures the use of indexed variables (which are themselves really just functions on the Natural Numbers).
- *Summation* – Summing the terms generated by the coordination of *iteration* and *function evaluation*. Interestingly, there may be two ways to think about this that are fundamentally different: you can either add as-you-go creating a running total, or generate the full list of terms and sum all-at-once at the end. This distinction is very important when you begin to consider infinite series and convergence.

Accurately interpreting a given summation-notation expression involves coordinating these mental activities or procedures:

For example, in order to expand

$$\sum_{k=2}^{7}(k+1)^2$$

Figure 1

one might go through the following mental actions (MA):

MA1. Recognition that the initial value of the index k is 2.
MA2. If $k <= 7$, coordination of the following actions:
(a) Action of evaluating the summand with the current value of the index (k).
(b) Action of adding the resulting value to the running total. Alternatively, the action of placing that value in a list to be added at the end.
(c) Action of incrementing the index to the next whole number.
(d) Return to Step 2 again.
MA3. Recognition that, after running through 2(a)-2(d) when $k = 7$, you are finished expanding the expression.

MA4. In the case of list generation, the final step would be to compute the sum.

After performing this sequence of actions one might interiorize that sequence and consider the whole expansion as a process. This process could then be conceived of as being constructed from the following sub-processes:

P1. Recognition that the initial value of the index k is 2.
P2. Construction of the process of k running through all of the successive integer values between 2 and 7.
P3. Coordinated with Step 2, construction of the process of the summand being evaluated over the range of k values.
P4. Recognition that these two processes together will generate a) a list of terms to be summed, b) an addition expression, or c) a number.

The process view outlined above represents a more sophisticated understanding of expanding summation notation, one that could be used to make sense of more complex ideas like Riemann sums. However, one must still go through the mental actions in order to actually generate the long-hand sum.

Expressing a Sum Using summation Notation:

Given the prompt,

Express the sum of the first five odd integers using Summation Notation,

one might go through the following mental constructions:

MC1. Construct, mentally or physically, the long-hand sum, e.g.:

$$1 + 3 + 5 + 7 + 9$$

MC2. Construct an indexing process
  a) Identify an appropriate starting value of the index.
  b) Identify successive integer values of the index with each term in the sum.
  c) Identify the appropriate terminating value of the index.

MC3. Construct a function that takes index values as its input and outputs the appropriate term of the sum. For this example, with index k, the function $(2k - 1)$ would generate the appropriate addends for integer values $1 \leq k \leq 5$.

MC4. Arrange the elements of the notation to indicate the desired sum.

We think of MC2 as outlined above as a kind of 'standard' way of constructing an indexing process. However, there are certainly non-'standard' ways of arriving at an equivalent end result. One might also first come up with a function that generates the odds and then adapt the indexing to generate the appropriate list to be summed. For instance, if you thought of odd numbers as one-greater than evens, you might want to use the formula $(2k + 1)$ to generate odd numbers. In that case, the next step would be to identify which values of k you would have to plug in to generate 1 as your first odd and 9 as your last.

Method

We designed a brief survey made up of three different tasks related to summation notation. The three tasks were designed to give students the opportunity to work with expanding a summation-notation expression and expressing a long-hand sum using summation notation. The first task was a context problem in which the students were asked to express a desired quantity using summation notation. There were two versions of this task, one whose solution involved using a composition of functions and one whose solution did not. With this first task we sought to simulate some of the challenges we noticed in our Advanced Calculus teaching experiment (Strand, Zazkis, & Redmond, 2012) *without* involving the area context. For the second task, students were asked to express the sum of the first ten odd integers using summation notation. For the third task, students were given a summation notation expression and were asked to write out the long-hand sum it represented. These pair of tasks were designed to investigate how well students could use and interpret summation notation in a simple context.

Task 1: In Physics class you and your lab partner have built a model roller coaster. In the table below, the average speed of the car (in meters per second) is recorded over each 1.5 second time interval. Using $\sum$-notation, express the total distance traveled by the car over six seconds.

Time (sec)	Avg Speed (m/s)
$0 - 1.5$	5.1
$1.5 - 3$	6.4
$3 - 4.5$	4.9
$4.5 - 6$	6.9

Figure 2

Task 2: Using $\sum$-notation, write an expression for the sum of the first ten odd integers.

Figure 3

Task 3: Write out the given summation longhand:

$$\sum_{k=2}^{7}(k+1)^2$$

Figure 4

The surveys were distributed to students in multiple sections of Calculus II (techniques and applications of integration), Calculus III (sequences and series), an introductory real analysis course, and a graduate-level analysis course. In this way we hoped to receive responses running the gamut of student experience with summation notation. We received 117 completed surveys, of which 98 were from the calculus sections.

The first round of analysis consisted of evaluating each survey response for correctness. During the second run-through the focus was on describing, as specifically as possible, each error that occurred. At this stage we were most interested in how errors on the simple tasks correlated with errors on the more challenging tasks. It was our initial attempt to explain the origins of the errors that led us to develop cognitive models of the mental activities involved in using summation notation.

Setting the data aside for the moment, we set about constructing the aforementioned cognitive models. These were developed a priori, drawing on our mathematical knowledge of summation notation and the constructs of APOS theory. After much thought and debate, we arrived at something we though might be useful for analyzing our data. Through the process of data analysis were able to further refine the models, until we arrived at something that was capable of explaining many of the most common and significant errors we were seeing .

Results

The cognitive models that we developed provide powerful tools for analyzing and explaining student errors when using summation notation. Due to length considerations, here we will briefly discuss some of the errors we were able to explain with the models.

A sample response to Task 3 is given in Figure 5.

$$(2 + 1)^2 + (2 + 1)^2 + (2 + 1)^2 + (2 + 1)^2 + (2 + 1)^2 + (2 + 1)^2 + (2 + 1)^2$$

Figure 5

This student seems to have been able to evaluate the function when $k = 2$. However, there seems to be no recognition of the indexing process (MA2c), and they appear to have taken the value of 7 (Figure 4) to be the number of terms in the sum. Here is a very static view of summation notation, with almost none of the underlying processes being demonstrated with this response.

A sample response to Task 2 is given in Figure 6.

$$\sum_{i=2}^{N}(i-1)$$

Figure 6

This student also provided an unprompted example that further illuminated their thinking (Figure 7).

$$\sum_{i=2}^{4}(i-1) = (2-1) + (4-1) + (6-1) + (8-1)$$

F

igure 7

While there is evidence of an iterative evaluation of the function (i – 1), the increment of the index appears to be 2. It is impossible to tell whether the *"i* = 2" is supposed to suggest an increment of 2, the starting value of the index, or both. Even the successful coordination of MC2 and MC3 does not guarantee successful adherence to the convention; the sub-process of incrementing the index by 1 when a different increment is desired presents a non-trivial challenge for many students.

A sample response to Task 1 is given in Figure 8.

$$\sum_{j=0}^{6}\frac{3}{2}f(x) = \frac{3}{2}(5.1) + \frac{3}{2}(6.4) + \frac{3}{2}(4.9) + \frac{3}{2}(6.9)$$

Figure 8

It seems reasonable that the student used "*f(x)*" to stand for the Avg Speed values from the table provided and that the 3/2 represents the time interval for each measurement. Notice that there is no indexed variable in the summand. We intentionally designed this problem so that there is no obvious rule that takes an index value to the list of average speeds (Figure 2). Thus it is not surprising that the student was unable to use an indexing process to generate the desired long-hand sum (which itself is a correct representation of the stated problem). However, the values for *j* going from 0 to 6 suggest that there may be further difficulties with MC2. While it is unclear exactly how they envisioned *f(x)* being evaluated, they did not demonstrate an ability to coordinate an indexing process with a function-evaluation process (MC2 and MC3).

Conclusions

The cognitive models outlined above provide a framework for analyzing student thinking about summation notation. Additionally, they could be used to design instructional tasks that would help students to develop an understanding of the underlying processes *and* ways in which they can successfully be coordinated (Weber, and Larsen, 2008; Asiala, Dubinsky, Mathews, Morics, & Oktac, 1997). It is worth noting, especially for educators, that summation notation encodes multiple potentially challenging processes for students. More than that, successful use of summation notation involves *coordinating* these challenging processes. The struggles we found with Task 1 suggest that it is likely that our original work in the context of area under a curve *revealed* difficulties with summation notation rather than *causing* them.

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