VENN DIAGRAMS AS VISUAL REPRESENTATIONS OF ADDITIVE AND MULTIPLICATIVE REASONING IN COUNTING PROBLEMS

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This case study explored how a student could use Venn diagrams to explain his reasoning while solving counting problems. An undergraduate with no formal experience with combinatorics participated in nine teaching sessions during which he was encouraged to explain his reasoning using visual representations. Open coding was used to identify the representations he used and the ways of thinking in which he engaged. Venn diagrams were introduced as part of an alternate solution written by a prior student. Following this introduction, the student in this study often chose to use Venn diagrams to explain his reasoning and stated that he was envisioning them. They were a powerful model for him as they helped him visualize the sets of elements he was counting and to recognize over counting. Though they were originally introduced to express additive reasoning, he also used them to represent his multiplicative reasoning.

Key words: Additive reasoning, Combinatorics, Multiplicative reasoning, Representations, Set Theory

Introduction and Research Questions

Piaget and Inhelder (1975) contend that children's combinatorial reasoning is a fundamental mathematical idea based in additive and multiplicative reasoning. However, the research indicates that students of all ages often struggle with counting problems (Batanero, Godino, & Navarro-Pelayo, 1997; Eizenberg & Zaslavsky, 2004; English, 1993; Hadar & Hadass, 1981). Though some studies have adopted counting problems as the backdrop for research in other aspects of student learning (Eizenberg & Zaslavsky, 2004; Fischbein & Grossman, 1997), very little research has been conducted on combinatorial reasoning. Shin and Steffe (2009) began to investigate how middle school students dealt counting problems based on their additive and multiplicative reasoning and determined levels of enumeration that appeared in the students' behavior: additive, multiplicative, and recursive multiplicative enumeration. Though they provide examples of students' visual representations of the elements being counting, Shin and Steffe (2009) do not focus on how students' reasoning can be expressed in their representations. Further, the problems their students encountered did not address operations more complex than permutations of distinct elements.

Research on Venn diagrams and their use in discrete math or probability courses seems to be of two minds. On one hand, Fischbein (1977) states that Venn diagrams are powerful models that can be used to solve a wide range of problems. Indeed, some combinatorics texts (e.g. Bogart, 2000) present Venn diagrams as a visual representation for basic counting problems. On the other hand, it has been reported that students have trouble using Venn diagrams and visualizing set expressions (Bagni, 2006; Hodgson, 1996) to the extent that some authors have recommended the removal of these representations from basic probability courses (Pfannkuch, Seber, & Wild, 2002) and some combinatorics texts (e.g. Tucker, 2002) introduce them only while solving complex counting problems. This study extends the current research by investigating combinatorial reasoning in relation to students' visual representations, particularly Venn diagrams. Thus, this study attempts to answer the following question: How could a student use Venn diagrams to express the additive and multiplicative reasoning he employs while solving counting problems?

Theoretical Framework

The primary tenant underlying this study is that mathematical knowledge is not received through the senses or communication but must be actively built up by the cognizing subject (Von Glasersfeld, 1995). According to Harel (2008), there are two categories of mathematical knowledge: ways of understanding and ways of thinking. The reasoning applied in a particular mathematical situation – the cognitive product of mental acts – is known as a *way of understanding*. On the other hand, *ways of thinking* refer to what governs one's ways of understanding and are the cognitive characteristics of mental acts. Ways of thinking are always inferred from ways of understanding.

The author developed a preliminary framework of students' ways of thinking about the set of elements being counted, known as the *solution set*, in basic counting problems (Halani, 2012a, 2012b). Three of these ways of thinking are relevant to this study and are summarized in Table 1. While engaging in the first, *Union* thinking, a student will first think globally and envision the solution set as the union of smaller subsets which he or she may believe to be distinct before taking the sum of the sizes of these subsets. The next two ways of thinking both involve answering a question that has not been asked. In *Deletion* thinking, a student will consider a related problem whose solution set contains a subset which is in one-to-one correspondence with the original solution set and then find an additive relationship between the sizes of the solution sets. In *Equivalence Classes* thinking, a student will consider a related problem whose solution set can be partitioned into equivalence classes, each of which correspond to an element of the original solution set, before finding a multiplicative relationship between the sizes of the solution sets. By their very definitions, Union and Deletion have their roots in additive reasoning and are therefore shown in orange in Table 1, whereas Equivalence Classes is multiplicative and shown in purple.

Way of Thinking	**Description**	**Example of a task whose solution could be driven by this way of thinking:**
Union	Envision the solution set as the union of smaller subsets. Add the sizes of the subsets.	How many 3-letter "words" can be formed from the letters *a, b,c,d,e,f* if repetition of letters is allowed and *d* must be used?
Deletion	Consider a related problem with solution set *C* which contains a subset, *B*, which has a bijective correspondence with the solution set of the original problem, *A*. Find an additive relationship between the the original and the new solution set.	How many 3-letter "words" can be formed from the letters *a, b,c,d,e,f* if repetition of letters is allowed and *d* must be used?
Equivalence Classes	Consider a given task with solution set *A*. Consider a related problem with a solution set *S* which can be partitioned into equivalence classes of the same size – each one of which corresponds to an element of *A*. Find a multiplicative relationship between the solution sets.	How many permutations of *MISSISSIPPI* are there?

Table 1: Some Ways of Thinking about Combinatorics Solution Sets

Methodology

Data for this study come from a teaching experiment (Steffe & Thompson, 2000) conducted at a large southwestern university in the United States. Al, a freshman enrolled in a second-semester calculus course, participated in nine teaching sessions with the researcher over a four-week period. Tasks for this study involved the operations of arrangements with and without repetition, permutations with and without repetition, and combinations. In addition, it is known that students do not always interpret combinatorial tasks in the same manner that the mathematical community does (Godino, Batanero, & Roa, 2005). As a result, tasks for this study were separated into two parts: a situation and a question (or questions). See Table 2 for the tasks discussed in this paper. Following the completion of many of the tasks, the researcher implemented the Devil's Advocate instructional provocation (Halani, Davis, & Roh, this issue) by presenting alternate solutions written by supposed previous students to Al for evaluation. He reinterpreted and justified the solution if he believed it to be correct, and refuted it if he disagreed. Through these alternate solutions, many visual representations such as tables, tree diagrams, Venn diagrams, slots, and mapping diagrams were introduced. Venn diagrams were first presented during the fourth session; however, set theoretic language was not employed. For example, the term "overlap" was used instead of a formal term such "intersection." The concept of a universal set was not introduced until the eighth session.

Session	Task	Statement
5	14(vi)	*Situation*: Suppose we have the letters *a,b,c,d,e,f* and we are forming three-letter strings of letters ("words") from these letters. *Question*: How many 3-letters "words" can be formed from these letters if repetition of letters is allowed and the letter "*d*" must be used?
6	16(iii)	*Situation*: A university decides that sorority names can be three-letters chosen from the following Greek letters: $\Gamma, \Delta, \Theta, \Lambda, \Pi, \Phi, \Psi, \Omega$. *Question*: How many sorority names can be formed from these letters if repetition of letters is allowed and the letter "Θ" must be used?
8	30(v)	*Situation*: Consider the word *WELLESLEY*. We will be forming "words" from these letters. *Question*: How many "words" can be formed from the letters in "WELLESLEY" if we need 4-letter words, each letter may be used multiple times, and we must use the letter "E"?
9	31(i)	*Situation*: Consider the word *MISSISSIPPI*. We will be forming "words" from these letters. *Question*: How many "words" can be formed from the letters in "MISSISSIPPI" if we need 11-letter words created by rearranging the letters provided?

Table 2: Relevant Tasks Implemented in the Study

There were a few phases of data analysis. Following each session with the student, content logs were created including summaries of the session along with observational, methodological, and theoretical notes (Strauss & Corbin, 1998). After the data were collected, each session was transcribed. Open coding (Strauss & Corbin, 1998) was used to identify the visual representations employed by the student. The student's ways of thinking and the type of reasoning in which the student engaged were also analyzed.

Results

After Venn diagrams were introduced during the fourth and fifth sessions of the study, Al demonstrated that he could visually represent his reasoning using Venn diagrams while engaging in the following ways of thinking: Union, Deletion and Equivalence Classes. As indicated in Table 1, the first two ways of thinking draw primarily on additive reasoning and the last draws upon multiplicative reasoning. He often employed Venn diagrams when asked to explain his reasoning and stated that he was envisioning them as he solved the tasks.

Venn Diagrams to Represent Additive Reasoning

Al recruited Venn diagrams as a tool to explain his additive reasoning employed while engaging in Union and Deletion thinking to solve counting problems.

Union. During the fifth session, Al was asked to complete task 14(vi) from Table 2. He first over counted and found that there were $3\times6\times6$ "words." Three alternative solutions were presented via Devil's Advocate (Halani et al, this issue),one driven by Deletion thinking and two which relied on Venn diagrams. In the next session of the study during a mid-study test, Al was asked to complete task 16 (iii) from Table 2. He first drew three sets of three slots and wrote 1 in the first slot of the first set, 1 in the second slot for the second set and 1 in the third slot for the third set. Each set of slots corresponded to a different subset of the solution set, based on the location of Θ. This indicates that he was engaging in Union thinking. His solution is shown in Figure 1. While determining the number of possible options in each case, Al was careful to avoid over counting by partitioning this union of sets. He multiplied the numbers in the slots for each set and then took the sum of these products to get 64+56+49.

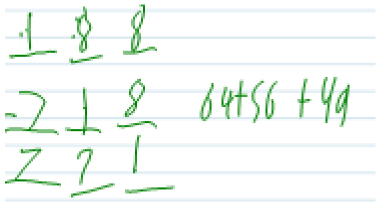

Figure 1: Al's solution to task 16(iii)

Al was asked about his confidence in his solution. In order to explain, Al referenced doing a similar problem during the fifth session and immediately drew a Venn diagram (not shown) to illustrate his additive reasoning. He explained his thinking:

"*I was trying to think, ok, we have each of these different I guess groups of where it can be. Like with this one I could tell that you have a group where it's* [Θ *is*] *the first letter, a group where it's the second letter, a group where it's the third letter* (draws three overlapping circles). […] *And I knew that for all of this* (indicates all of the first set), *I can only count this much of this* (indicates the elements in the second set excluding the first set), *and I can only count this much of this* (indicates the elements in the third set which have not yet been counted)."

Even though Al did not draw a Venn diagram during his counting, it seems as if he may have been envisioning one from his explanation. It is clear that while he was counting, he was attending to the intersection of the subsets based on the location of Θ. The first Venn diagram Al drew was hard to read so Al drew a second one (see Figure 2). When Al was asked to compare his current thinking about this type of problem to the similar problem he said they had encountered in the previous session, his response follows:

"*Well, I think before, I would list them all, or I guess I didn't have as clear of a way of understanding that repetitions occur in this type of problem.* […] [*Now*] *I'm using some way to define what these three sets are. And I'm defining* […] *the first set as places where the first variable is theta. Defining that group* (points to second circle in Figure 2) *as where the second variable is theta, and that group* (points to third circle) *where the third variable is theta. And by defining them, I guess I was kind of realizing that they overlap when both the first and the second requirements are met. Or when the first and the third. Or when all three are met. So by kind of knowing that the only place I'm going to have* *repetitions is where that's true and that's true* (points to intersections of two sets), *or when all three are true, then I could kind of look for it better*."

Here, it is clear that he was envisioning this Venn diagram even though he did not originally visually represent his reasoning while solving the task. From his comparison of his current thinking to his previous thinking, it appears as if Venn diagrams helped him clearly picture what he was counting so that he was better able to avoid over counting.

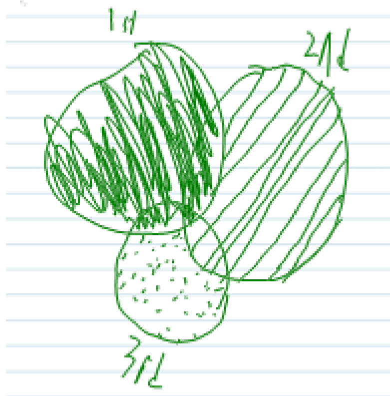

Figure 2: Venn diagram for Union thinking for task 16(iii)

Deletion. Al employed two different variations of Venn diagrams to represent his Deletion thinking – one in the fifth session and a second during the eighth.

Session Five. Devil's Advocate (Halani et al., this issue) was used to provide the following argument for task 14 (vi) – it is driven by Deletion thinking and written by a supposed former student, Carrie: "We first determine the number of 3-letter 'words' possible regardless of whether '*d*' is used: $6\times6\times6$. Then, we determine the number of 'words' which do not include '*d*': $5\times5\times5$. Thus, there are $6^3-5^3=91$ 'words' which include the letter '*d*.'"

Al was asked if he had seen an argument like Carrie's before. He had naturally engaged in Deletion thinking for previous problems; however, his response refers to Venn diagrams:

"*It's kind of like the Venn diagram but it's kind of not.* […] *It's kind of like the Venn diagram, cause in the Venn diagram you have kind of these two circles* (draws the two circles in Figure 3), *but she was saying that is with* 'd' (writes "*d*" in the portion in the right circle that is not in the left circle) *and then this is with all the possibilities without* 'd' (writes "~~d~~" in the portion in the intersection of the circles). *So she just kind of ignored this* (scribbles in the portion of the left circle that is not in the right circle)...*this is all the possibilities with* 'd,' (indicates the entirety of the right circle) *then she subtract[ed] the* [*possibilities*] *without a* 'd' *to figure out how many just have* 'd'"

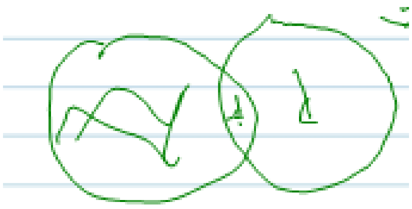

Figure 3: Venn diagram for Deletion thinking from session 5

At this point in the study, Venn diagrams had been introduced to solve other questions involved in task 14, but they all involved two sets with a non-empty intersection. As mentioned in the previous section, the concept of a universal set had not been introduced. Thus, Al's Venn diagram for Carrie's reasoning was based off the Venn diagrams he had seen before and therefore involved two sets with a non-empty intersection. His representation for Carrie's Deletion thinking involved counting everything in the right circle of Figure 3 and then subtracting the number of elements in the intersection. Thus, it seems as if Al understood that Carrie constructed a new problem (that of determining the total number of 3-letter words) and then found an additive relationship between the new solution set and the original solution set.

Session Eight. In the eighth session, Al tried to solve task 30 (v) from Table 2. At first, he over counted and found the answer to be $\binom{4}{1}\cdot 5^3$ because he considered places the E could go and then determined that there 5 choices for each of the remaining spots. The researcher reminded Al that he should ensure that he had counted everything he wanted to count and that he had not counted the same thing more than once. He quickly realized his mistake and determined the solution to be $5^3+5^2\times 4+5\times 16+4^3$ by engaging in Union thinking with subsets determined by the location of E and then carefully ensuring he does not over count the intersections of these subsets. He explained that it reminded him of the "*Venn diagram problem and that kind of whole picture* (draws Venn diagram with 4 overlapping circles) *just popped into my head*." Once again, it is clear that he is envisioning a Venn diagram for Union thinking although he did not draw it while counting.

The researcher reminded Al of Carrie's argument for task 14 (vi). Al said, "*So in this case, it would be* 5^4-4^4 *.*" At this point, the researcher introduced the Venn diagram shown in Figure 4. She explained that the box represented the whole universe that they were concerned with. She then sketched the four circles representing subsets based on the location of E. The researcher asked Al what was actually being counted. Al said that the entire box was being counted and then everything that was not in the circles was being subtracted. As before, Al demonstrated that he could use Venn diagrams to represent his additive reasoning employed while engaging in Deletion or Union thinking.

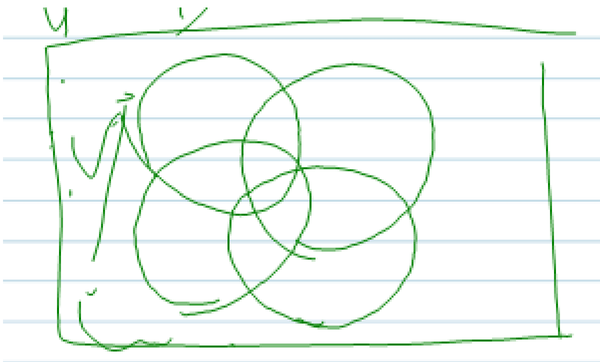

Figure 4: Venn diagram for Deletion thinking from session 8

Venn Diagrams to Represent Multiplicative Reasoning

Mapping diagrams were introduced as visual representations for Equivalence Classes thinking, yet Al never employed them himself. Instead, Al seemed to make a connection between Deletion and Equivalence Classes thinking and began used Venn diagrams with a universal set to represent his multiplicative reasoning in the latter case. This seems to be an example of actor-oriented transfer (Lobato & Siebert, 2002). In the last session of the study, the researcher asked Al to give some examples of visual representations. His response regarding Venn diagrams is below:

> "*There's been kind of Venn diagram style overlap* (draws the Venn diagram with a rectangle and three circles shown in Figure 5) *and then there's been kind of a way that you could also figure that out by taking the whole* (indicates entire rectangle) […] *and then you're dividing out* […] *this kind of bad area* (shades in the complement of the three

circles) [...] *Because when it comes to situations with* […] *a lot of different overlaps* […] *like if there's a fourth circle* (draws the fourth circle in the figure) […] *then it'll get kind of complicated and so it would almost be easier to kind of find the whole thing and then kind of take out the stuff you don't want* […] [*by dividing*] […] *You figure out the ratio."*

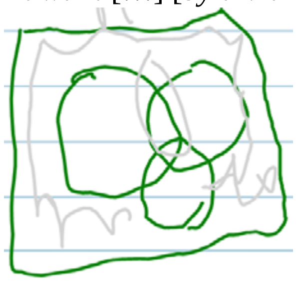

Figure 5: Venn Diagram for Equivalence Classes thinking

To Al, the universal set in Figure 5 is the solution set to a different problem, one which involves things that he wants represented as the union of the circles and things that he doesn't want. In the previous session, Al determined an additive relationship between the solution set of the original problem and that of the new problem. In this very general case, Al can imagine a multiplicative relationship existing and using this ratio to solve the problem.

Al then demonstrated his use of Venn diagrams for multiplicative reasoning. He had previously engaged in Equivalence Classes to solve task 31(i) from Table 2 in order to determine that there are $\frac{11!}{4!\cdot 4!\cdot 2!}$ permutations of the letters in MISSISSIPPI. He explained that there were 11! ways to permute 11 distinct objects and drew a rectangle to represent these 11! elements. He then drew an oval in this rectangle and wrote "*g*" for "good" inside it. He explained that for each "good" thing there were 4! ways to rearrange the Ss, 4! ways to rearrange the Is and 2! ways to rearrange the Ps, while shading in the complement of the set "*g*." He stated, "*I knew if I were to attempt to try to find what's inside the 'g' by itself, it's kind of hard. But I realized that if I were able to find everything* […], *it would be a bit easier*." Thus, he was visualizing a Venn diagram to explain the multiplicative reasoning he employed while engaging in Equivalence Classes.

Discussion

This study is a step towards better understanding the connection between student reasoning and visual representations as they solve counting problems. The data indicate that Venn diagrams were a powerful model for Al – he often stated he was envisioning them as he was counting and they helped him avoid and recognize over counting. He employed Venn diagrams to represent both his additive and multiplicative reasoning. In fact, he transferred the idea of a universal set to Equivalence Classes thinking to represent his multiplicative reasoning even though mapping diagrams, not Venn diagrams had been introduced for that way of thinking. Because of the similarities between Deletion and Equivalence Classes, it may be useful for students to see the same type of representation for both despite the difference in additive versus multiplicative reasoning.

The results of this case study support the inclusion of Venn diagrams in the combinatorics or basic probability curriculum as early as the use of arrangements with repetition. Indeed, it seems as if introducing Venn diagrams could push students to become more cognizant of over counting and recognize how to correct these types of errors. Further, it could be helpful for teachers to introduce the concept of a universal set when engaging in Deletion or Equivalence Classes in order to help students build connections between the solution set of the new problem with that of the original problem. Finally, Venn diagrams may help students explain their additive and multiplicative reasoning, just as they did for Al.

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