ON THE EMERGENCE OF MATHEMATICAL OBJECTS: THE CASE OF e^{az}

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In this report we propose an alternate account of mathematical reification as compared to Sfard's (1991) description, which is characterized as an "instantaneous quantum leap", a mental process, and a static structure. Our perspective is based on two in-service teachers' exploration of the function $f(z)=e^{az}$, using Geometer's Sketchpad. Using microethnographic analysis techniques we found that the long road to beginning to reify the function entailed interplay between body-generated motion and object self-motion, kinesthetic continuity between different sides of the "same" thing, cultural and emotional background of life with things-to-be, and categorical intuitions. Our results suggest that perceptuomotor activities involving technology may serve as an instrument in facilitating reification of abstract mathematical objects such as complex-valued functions.

Key words: Complex variables, Intuition, Perceptuomotor, Reification, Technology

Background

Sfard (1991, 1994) is one of the leading contributors to the theory, which describes how one navigates from an operational (process) conception to a structural (object) conception of the same mathematical notion and identified three stages of this development, which are *interiorization*, *condensation*, and *reification*. During the *interiorization* stage one is able to perform a process on a familiar object and thus, the process becomes a mental entity. As an example, Sfard portrays students who are proficient at taking square roots to be in the interiorization stage for conceptualizing complex numbers. *Condensation* is the second stage and it occurs when one is able to view a process as a whole and compact entity. For example, students who perceive $5+2i$ as a shorthand for certain procedures would still be able to use it in multifaceted algorithms. The third stage, *reification*, is present when one identifies a novel entity as an object-like whole. Learners who are at this stage would recognize $5+2i$ as a legitimate object and member of a well-defined set. According to Sfard,

> Only when a person becomes capable of conceiving the notion as a fully-fledged object, we shall say that the concept has been reified. *Reification*, therefore, is defined as an ontological shift – a sudden ability to see something familiar in a totally new light. Thus, whereas interiorization and condensation are gradual, quantitative rather than qualitative changes, reification is an instantaneous quantum leap: a process solidifies into object, into a static structure. (1991, 19-20)

While Sfard's theoretical framework is frequently adopted for investigating mathematical reasoning, in this preliminary report we propose an alternate account of mathematical reification, which is neither "an instantaneous quantum leap," nor a matter of mental processing, nor a static structure. We describe reification as a long, never-ending, embodied process that grows organically out of perceptual and motor practices. We draw a parallel between the common experience of an object or type of object coming to be and a mathematical object coming to be. Our perspective is based on results from the research question: How do in-service teachers determine the behavior of the function e^{az+b}, after working with the function e^{z}?

Theoretical Perspective

The question of how a thing comes to be in the course of human experiences is a very old one in philosophy. All perspectives in philosophy have elaborated on ideas that relate to it in different ways, across ontology (i.e. the investigation into what kinds of entities exist, and how they are organized in classes) and epistemology (i.e. the investigation into how we come to know that which exists, and what different forms of knowable existence are). When the entities in question are mathematical (e.g. exponential functions), questions of existence and "thinghood" lay at the center of the philosophy of mathematics. A review of the relevant literature falls necessarily outside of the scope of this paper. Thus, we limit ourselves to asserting that we adopt those crucial experiential aspects of how physical objects come to be, as a guide into the emergence of mathematical objects. Specifically, we identify three major interwoven aspects that participate in the advent of a physical thing: (1) perceptuomotor activity, (2) cultural and emotional background of life with things-to-be, and (3) categorical intuitions, or the evolving intuitive sense for categories of objects.

We cite certain works of Husserl as inspiring investigations on these three aspects. Regarding perceptuomotor activity, Husserl (1907/1997) discussed the interplay between visual and tactile "spaces" in the constitution of a thing, and how these spaces get traversed and meshed by motor activity and kinesthesia (i.e. sensations of movement and of muscular activity). In this regard, Husserl elaborated on two issues that we will use for our present study: a) the interplay between body-generated motion and object self-motion (e.g. to catch a falling ball we compose our bodily running to a certain target location, while we visually track the ball as it falls "on its own"), and b) the kinesthetic continuity between different sides of the "same" thing (e.g. two sides are of the "same thing" to the extent that we can transition visually and tactually *continuously* from one to the other). Cultural and emotional background of life with things-to-be encompass our life-histories with objects, endowing them with emotional-cultural values (e.g. a certain atlas is the one my father gave me as a gift when I was a child). Husserl (1970a) proposed that these emotional-cultural histories, which attach themselves to objects, symbols, and others, form the "lifeworld" in which we live. Finally, Husserl (1970b) proposed that we perceive not only individual entities but also categories that subsume them. In other words, that the notion of a class is not derived from an intellectual grouping of individual instances — the latter presumed to be the only ones perceptually accessible — but of direct intuitive grasp. A useful example to better understand this idea is the situation of looking for something of a certain color; we are told, that a missing book "is blue" and in our efforts to find it we identify all the blue things in sight. This quality of blueness is not of a specific tone of blue unique to the missing book; rather, we survey our surroundings from which a certain color-based class of books becomes prominent and segregated in perceptuomotor ways. Capturing such interplay and continuity requires meticulous analysis, which we discuss in the methods section.

Methods

The participants in this study were enrolled in a technology course designed for pre- and in-service secondary mathematics teachers. The instructor taught the course in a situated learning environment, with a focus on complex numbers. Using *Geometer's Sketchpad (GSP)*, the students made connections between the Cartesian and polar form of complex numbers, explored the dynamic aspects of the arithmetic of complex numbers, discovered the behavior of the functions $f(z) = az + b$, $f(z) = z^2$, and $f(z) = e^z$, and examined the geometric aspects of the complex roots of a quadratic equation. A hallmark of the course was that the students' questions and discoveries served as a source for further activities. Our data for this report is a result of such a discovery made by Brent and Rick (both pseudonyms) as they explored the behavior of the function $f(z) = e^{az+b}$ using *GSP*, during an interview. We incorporated microethnographic analysis techniques (Erickson, 1996) in our research, which demands detailing moment-by-moment, audible, and visible human interactions as they occur naturally within a given context. In *Elan*, we synchronized the video with the *GSP* screenshots and documented the moment-by-moment gestures, facial expressions, utterances, and reactions, as the students interacted with one another and their *GSP* production.

In exploring the function $f(z) = e^{az+b}$, Brent and Rick simplified the task by focusing on e^{az} and created a motion controller for the parameter *a*. In our selected episode, the participants attempt to determine why the curve for e^{az} collapses to a circle. From this episode we garnered 2.33:417 minutes of data. As part of our results we provide detailed descriptions of selected moments accompanied with video clips and commentary supporting our hypothesis. We used the following coding system in our analysis: numbers in parenthesis indicate the length of a pause in seconds, bold indicates overlapping speech, and double parenthesis and italics indicates a gesture description. The B signals Brent's utterances and the R signals Rick's utterances.

Results

We exemplify four moments from the interview that implicate the constitution of e^{az} as an object. The four moments entail: interplay between body-generated motion and object self-motion, kinesthetic continuity between different sides of the "same" thing, cultural and emotional background of life with things-to-be, and categorical intuitions. Before describing these moments, we provide a backdrop for the reader. The motion controller, created by Rick, enabled the participants to compare e^z and e^{az} where $z = -.5 + .96i$ and $a = 1.68 + yi$ with $-1.5 \le y \le 2.2$. Prior to our selected moments, the participants are surprised by the various behaviors of e^{az}. Rick finds it humorous that as the imaginary component of *a* approaches -1.2 the curve "flattens", and is intrigued that the curve gets "tighter" as the imaginary component gets larger and creates more spirals going counter clockwise. Both participants are mesmerized as the spiral transforms into a circle and suddenly the circle unwraps into a spiral going in the clockwise direction. They describe this instant as "cool", "crazy" and "the twilight zone".

In an effort to better understand the "something special" that occurs at this instant, Rick stops the motion controller in order to drag *a* manually and determine the exact value of *a*, that produces the circle. This is the moment where we observe interplay between body-generated motion (Rick dragging the point *a*) and object self-motion (spiral becoming a circle). With both hands on the mouse pad, Rick looks at the screen as he toggles the pointer around the point *a*. As Rick drags the point *a* down the vertical line, the spirals begin to collapse onto one another to form a tighter circle. Below is the dialogue that occurred, with descriptions of gestures, along with Figure 1, which contains video-clips and *GSP* screenshots from this exchange.

R: There's something special about this value right here ((*Rick has both hands on mouse pad as he looks at the screen and toggles the pointer around the point a. The point a moves down the vertical line and the circle become tighter*.))

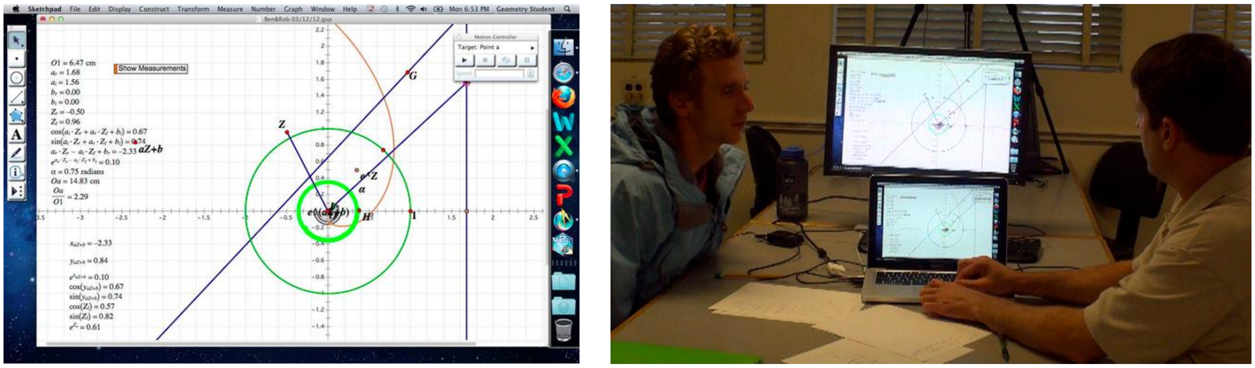

Fig. 1a. Circle circumference thicker   Fig. 1b. When it comes back on itself

Fig. 1. Something Special

B: Which value you looking at? ((*Brent brings arms down and leans slightly forward to look at screen. Rick continues to work on the mouse pad with gaze towards the screen and toggles the point a so the circumference of the circle is thicker*)) (See Fig. 1a)

B: oh when it comes kind of back on **itself** ((Brent *leans in forward to where his chest touches the edge of table as he looks at Rick who begins to lean back. Rick continues with same hand position on mouse pad causing the circumference of the circle becomes thinner*)) (See Fig. 1b)

R: **Ya** it's ya

R: **and it's** that's just ((Rick *continues to slightly toggle the pointer around the point a so that the circumference gets thicker and thinner while he looks towards the screen. At end of episode he slightly lifts his right hand off the mouse pad*))

B: **and then springs back out** ((Brent *leans back slightly, puts hands on chair and lightly lifts himself up as he turns toward the screen*))

This instant when the spiral "comes kind of back on itself … and then springs back out" prompted the participants to explore the possible origins of such behavior. During this exploration the participants discover that the curve of e^{az} becomes a circle if $a = x + yi$ and $\frac{x}{y} = 1.09$. This finding provoked the participants to investigate whether they obtain the same result when $x < 0$, which required dragging their vertical line so that it intersected the negative real axis. This moment could be considered as interplay between body-generated motion and object self-motion, but we also interpret it as kinesthetic continuity between different sides of the "same" thing, because the participants are investigating the same thing but from a different angle. As Rick drags the vertical line towards the negative real axis, he exclaims, "it's doing the same thing but in reverse". During this utterance, both participants look towards the screen as the camera zooms to the screen.

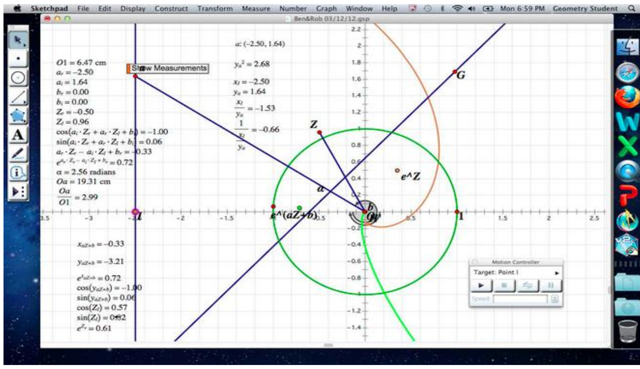

Fig. 2a. x-intercept of line is -2.5

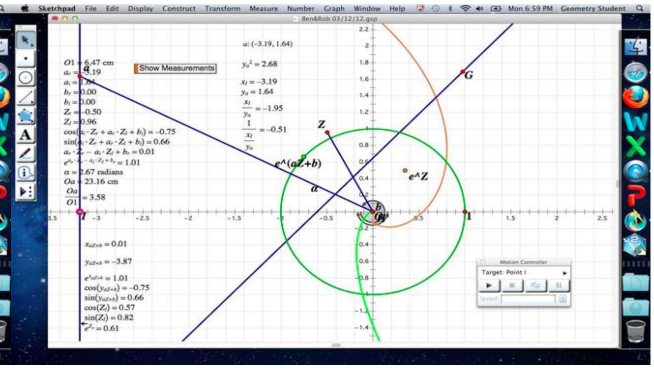

Fig. 2b. x-intercept of line is -3.9

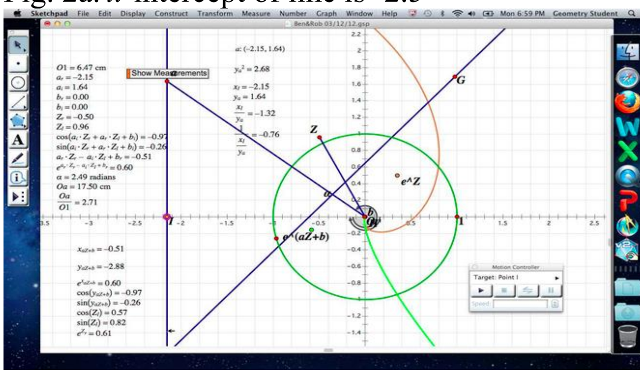

Fig. 2c. x-intercept of line is -2.15

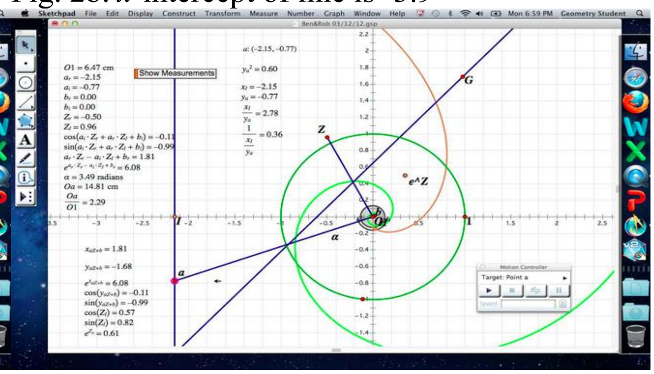

Fig. 2d. Point a = (-2.15, -0.77)

Fig. 2. Same thing but in reverse.

The *GSP* screenshot started at x=-2.50 (See Fig. 2a) and then Rick drags the vertical line using the point I (x-intercept of the line) to the point x=-3.19 (See Fig. 2b) and then brings it back to the point x = -2.15 (See Fig. 2c) where the curve of e^{az} is in the 4th quadrant but looks a bit flat. At this point, the curve of the function does not look like a spiral, which might have motivated Rick to drag the point a to (-2.15, -0.77), where the curve begins to take on the shape of a spiral (See Fig. 2d) and Brent is eager to see when the spiral will collapse onto itself. The dialogue and behavior below further illustrates that the participants fully anticipate obtaining the "same" thing for negative x-values.

B: see when you can collapse it in on itself ((*Brent has hands to his side, sitting back and looking at screen. Rick has his hands on the mouse pad and is looking at the screen. Camera starts to zoom into the screen at the point that Brent says 'when'. Initially the screenshot contains a spiral going counterclockwise and a =(-2.15,-1.84). Rick drags the point a up the vertical line to the point (-2.15,-.76) so that less of the spiral is visible.*))

(1.553) ((*Rick drags the vertical line to x=-2.05*))

B: like when we were doing ((*Rick continues dragging the vertical line to x=-1.35 and more spirals start to emerge going counterclockwise.*))

(3.900) ((*Camera is still zoomed in on the screen and starts to zoom out as Rick drags the point a down the vertical line and quickly bypasses the point where the circle appears and gets the spiral going clockwise. Both participants maintain their stance of looking towards the screen.*))

The participants' interactions with *GSP* did not fail them in generating a circle where the real component of a was negative, but it was not the circle they expected. The verbiage below highlights a moment where the participants' cultural and emotional background of life with things-to-be is accentuated. Although both Brent and Rick anticipated obtaining a circle, because of their experience with e^{az}, where the real component of *a* was positive, they were not prepared to see a circle of radius greater than one. Their familiarity with the behavior of e^{az} did not prepare them for the pleasant surprise that they observed on the screen – it seemed that they were anticipating a circle with radius less than one. They were also amazed to find that the circle appeared for the same ratio of $\frac{x}{y} = 1.09$.

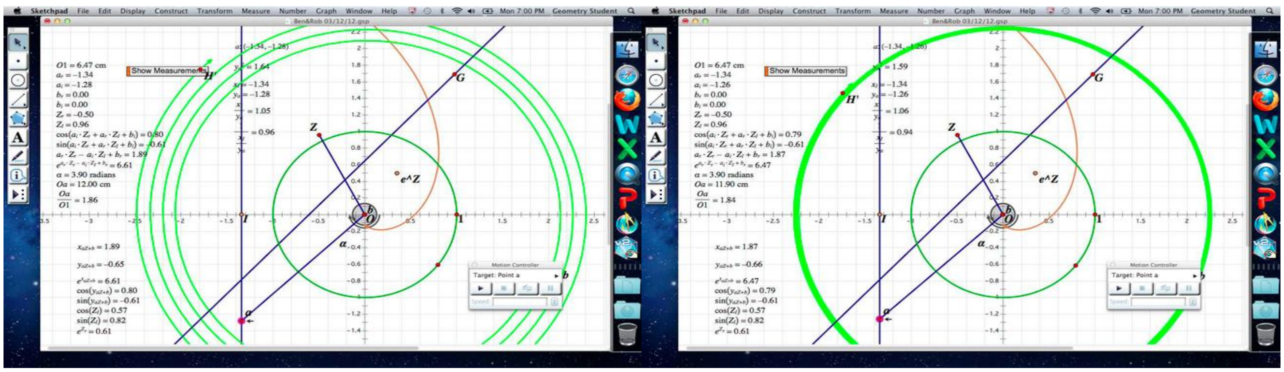

Fig. 3a. Huge   Fig. 3b. Wow

Fig. 3. Circle with $x < 0$.

R: Hu**ge** ((*Rick drags the point a and the concentric circles appear to be getting closer to one another and their radii are bigger than one*)) (See Fig. 3a)
B: **Wow** ((*Rick toggles the point a so that the concentric circles become one circle at the point (-1.34,-1.26). Both participants maintain their stance*)) (See Fig. 3b)
R: Huuuge ((*GSP screen shot is the same and both participants are still looking towards the screen as the camera starts to zoom out*))
(1.356) ((*camera zooms out*))
R: oh and ((Rick *toggles the point a up and down to get a slightly tighter circle for the value a = (-1.34,-1.26)*)) **the ratio we have**
B: **and it's** still the same ((*looks towards Rick*)) ratio ((*turns back to look at screen*))

The three moments described thus far, center on the senses i.e. seeing the behavior, dragging points, and feeling anticipation and excitement. For the most part these three moments are intertwined – that is, it is possible to observe interplay between body-generated motion and object self-motion, kinesthetic continuity between different sides of the “same” thing, and cultural and emotional background of life with things-to-be as the participants reason about the function e^{az}. Our fourth moment related to categorical intuitions, is slightly distinct from the others in that it centers on the algebraic notation used to treat the curve on the screen as a particular class of functions characterized by an algebraic form: e^{az}. This occurrence was prominent when the participants attempted to explain why the curve became a circle.

R: that’s just this ((Rick *points to the expression on his paper with his pencil, his right pointer finger is stretched out on the pencil as his left forearm rests on the table slightly parallel to the table and his knuckles touch his right forearm which is outstretched so that the two make about a 45 degree angle. Brent leans forward with his hands clasped together and his arms lying on the edge of the table as he gazes towards Rob's paper. On GSP the point a is initially not visible because it is under the motion controller frame. The spiral looks like* *concentric circles spiraling clockwise. As a appears, the spiral becomes a circle and starts spiraling counter clockwise*)) (See Fig. 4a)

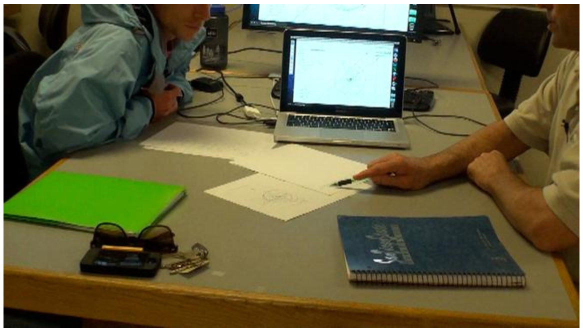

Fig. 4a. Connecting screen output to algebra

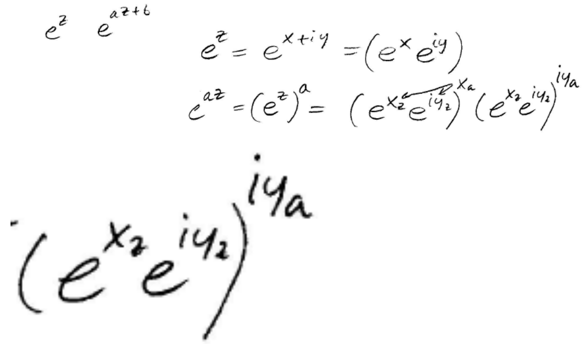

Fig. 4b. Rick's algebra

Fig. 4. Algebraic notation

R: guy right here ((*Rick starts to pull his pencil away and the camera zooms into his work. On GSP the point a continues to move down the vertical line and less spirals appear on the screen*))
R: and that's ((Rick *points to the same piece of the expression using his index finger, while the point a continues to move down the vertical line*))
R: because it's taking thee ((*taps the last factor of his expression* (bold part of algebra work shown in Fig. 4b) *twice with his left index finger as he gazes forward. Brent is still leaning forward looking at Rick's paper and then turns his head left towards the computer as Rick says "thee". The point a traverses the vertical line down and gets below the x-axis.*))
R: cosine and sine of ((*While leaving his left index finger on his paper he turns towards the screen and then towards Brent as he says the word of. The point a is almost off the screen at the bottom of the vertical line and the green curve is close to intersecting the point G.*))
R: the y-value ((*looks back down at paper while Brent stays in same body position but turns his head right to look at Rick's work. On GSP the point a is traveling up the vertical line.*))

The participants' attempt to reason about the curve by thinking about the algebraic pieces that make-up the function and the role that each piece plays may not be surprising given, Sfard's (1991) stages to reification begin by performing a process on a familiar object. What is more telling is that the participants did not do this in isolation – thinking about the process entailed comparing their algebra in conjunction with the dynamic aspects of the function itself.

Discussion

In the end the participants were not able to explain why the curve of the function e^{az} collapsed to a circle, but they were on a road to reifying the function e^z. Our evidence suggests that this road did not consist solely of mental processes or working with static structures. Instead it entailed an embodied process resulting from their prior experiences as they engaged in perceptual and motor practices with *GSP*. These results may imply that perceptuomotor activities are a vital component in developing an object-view, a "thinghood" perspective of abstract notions. There are various ways how such activities can be integrated into the classroom, but it appears technology is an excellent instrument for reifying complex-valued functions.

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