Coherence from Calculus to Differential Equations

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Introduction Recent reform movements, including the Mathematical Association of America's Curriculum Foundations Project, have sought to make calculus courses more coherent with other fields (Hughes Hallet, 2000). Despite this and other interdisciplinary reform efforts, post-calculus instructors still express extreme dissatisfaction with their students' mathematical preparation (Ferguson, 2012). We do not downplay the importance of the lamentations of other disciplines, but instead suggest that there also exist coherence issues within mathematics. Our hypothesis is that there is a mismatch between how students are expected to know calculus at the end of a course and how students are required to use calculus in future courses. Specifically, we examine calculus knowledge for differential equations.

Background Learning differential equations (DEs) is difficult and that difficulty is compounded by the multitude of other areas also needed, such as science, lifelike situations, and complex mathematics. DEs classrooms have benefitted from successful school mathematics approaches applied in a college classroom context (e.g., O. Kwon, Allen, & Rasmussen, 2005; Rasmussen & O. N. Kwon, 2007; Czocher & Baker, 2011). Past research has shown that there are foundational ideas from pre-calculus and calculus that are vital to success in a DEs course. Such ideas include: equation and solution (Raychaudhuri, 2008), mathematical differences among quantity, rate and rate of change (Rowland & Jovanoski, 2004; Rowland, 2006). Despite the existence of such foundational ideas, their coherence from calculus to DEs has not been explored.

Beginning from the calculus reform movements of the 1980s, calculus has been studied in many different contexts from many different research angles. Calculus instruction has benefited from taking advantage of contemporary theories specific to mathematical thinking. Many aspects of the calculus reform the emphasis on conceptual understanding and multiple representations, together with increased use of technology since the early 1990's are so embedded in calculus practice that they are considered mainstream (Ferrini-Mundy & Gucler, 2009). We suspect that these instructional advances cannot be utilized to their fullest without examining how the calculus must be known in future courses and in our case, in DEs.

Our purpose in this research was to examine how expected understandings of calculus topics align with the expected understandings of these topics in later courses. fThat is we examine if and how calculus and DEs align epistemically.

Research Frameworks In this work, we adopted the Calculus Content Framework (CCF) to examine important calculus concepts and skills (Sofronas et al., 2011). The CCF was built to organize goals from the first year of calculus into four strands: the first strand is sufficient for our purposes as it covers mastery of fundamental concepts and skills. The concepts are: derivative, integral, limits, approximation, sequences and series, Riemann sums, parametric and polar equations, continuity, and optimization. The skills are: derivative computations, manipulating algebraic expressions, area and volume, parametric equations, polar coordinates, trigonometric manipulations, facility with logarithms and exponentials, epsilon-delta proofs, listening and reading comprehension, facility with definitions and notation.

To meet our theoretical needs, we draw from parts of APOS theory (Dubinsky & McDonald, 2001), conceptual analysis (Thompson, 2002; Thompson, 2008), and didactical phenomenology (Freudenthal, 1983). Though APOS theory decomposes a concept into four stages (Action-Process-Object-Schema), it does not account for multiple concepts being utilized within the same mathematical setting. Didactical phenomenology and conceptual analysis suggest how a learner might sequence a mathematical idea, but is not embedded in a mathematical setting that a student might encounter. We required a theoretical perspective that could account for connections among concepts and that could be situated within the kinds of mathematical settings that are described in syllabi.

Combining pieces of all these frameworks, we created a new technique of analysis we termed *mathematics-in-use* which builds on Freudenthal's 1983 position that mathematical objects are created as an organizational scheme for mathematical phenomena. Our technique examined, through reflective reading of the texts and paradigmatic exercises, conceptual mathematical prerequisites, how these multiple mathematical concepts come together, how they are used, and how understandings of them may shift in order to structure a mathematical setting (task, example). Mathematics-in-use is an in-depth examination of how mathematical objects (concepts, ideas) must be interpreted for the problem solver to use them within the context of an example or task. We explore applying mathematics-in-use to different aspects including calculus concepts as a whole and in worked exercises. We believe that the best way to share our intentions for mathematics-in-use as a method of analysis is through an example, however that is beyond the scope of this current paper. We will focus on the calculus concepts in this paper, but a detailed examination of a worked exercise is in press (Czocher, Tague, & Baker, in press).

Methods In working with the above frameworks, we needed to use a multistage approach. Ferguson (2012) used a similar approach to examine how other disciplines required their students to know calculus. First, identify the topics and skills vital to DEs. Second, clarify the calculus topics and skills needed for successful understanding of the DE topics and skills. Third, apply our technique of mathematics-in-use to reflect on the process of solving exercises.

Before we describe our methods, we note that the differential equations course we draw from in this study is specifically for scientist and engineers. Since approximately 95% of the students in first-year mathematics courses are not majoring in mathematics (Ganter & Barker, 2004), our choice is most certainly warranted.

In the first stage of our multistage approach, we examined two textbooks currently in use at The Ohio State University (OSU). The textbooks are described in greater detail in some of our previous work (Czocher & Baker, 2011), but we offer some brief descriptions here. *Elementary Differential Equations and Boundary Value Problems* is a common differential equations textbook within the United States (Boyce & DiPrima, 2009). The overall emphasis of the book is on analytic techniques and it is organized by solution type. *An Introduction to Differential Equations for Scientists and Engineers* was written specifically to meet the needs of scientists and engineering majors at OSU (Baker, 2012). The organization of the book centers around exemplary problems from science and engineering and approaches those problems through mathematical modeling. Linear algebra is not a prerequisite for DEs at OSU, and so neither book requires prior linear algebra knowledge. However, both books assume previous knowledge of multivariate calculus as well as sequences and series. We examined the textbook table of contents, homework assignments, course objectives, and syllabi. Lastly, we interviewed five engineering faculty members who were teaching courses that listed DEs as a prerequisite, two mathematics faculty members, and five teaching assistants assigned to a differential equations course.

In the second stage, we used the CCF (Sofronas et al., 2011) to identify the calculus topics and skills necessary to explain the DE topics and skills. Some additions to the framework were necessary, for example, the derivative computations topic needed to include partial derivative computations. We also added the fundamental theorem of calculus topic as it is vital to differential equations in explaining the relationships between derivative and integral.

Figure 1 offers a pictorial view of our decomposition results. It is a matrix with DE topics displayed horizontally across the top and calculus topics (via the CCF) vertically down the left side. The top panel of the left side includes calculus concepts and the bottom panel includes calculus skills. Where dots are present indicates that a particular calculus concept or skill is necessary for the above differential equation concept. For example, in the column labeled separable equations, the calculus concepts integral and fundamental theorem as well as the calculus skills integration techniques, algebraic expressions, trig manipulations, and logarithms and exponentials are necessary.

Using mathematics-in-use, our decomposition shows a set of epistemological mismatches between calculus and differential equations. These mismatches are examined more fully in the next section, summarized in Table 1, and an extended example is available in our submitted manuscript (see Czocher, Tague, & Baker, in press).

Content Analysis The matrix from Figure 1 provides a visual representation of the ways that differential equations content relies on calculus concepts and skills. Horizontally, across the top, are the DE topics and vertically down the left side are the calculus concepts and skills identified from the CCF (see Sofronas et al., 2011) with our necessary additions. If a dot appears in box i,j, then it signals that the calculus content in row i is necessary for the differential equations topic in column j.

Similar to Ferguson's 2012 study of calculus, the suggestions of faculty members for topics covered in differential equations were dependent on whether the individual was a teacher of mathematics or a user of the mathematics. Thus, only one engineering faculty member chose existence/uniqueness as crucial, whereas all mathematics faculty members listed it as vital. In cases such as these where there was consensus among an entire group, we included the topic. Some professors also suggested Laplace transforms as vital, however, this topic is not included in the introductory differential equations course for engineers and scientists, and as such it was not included in our content analysis.

It is evident from a quick glance at Figure 1 that there are far more marks in the bottom panel than in the top panel, leading to the conclusion that DEs is a skill-based collection of topics. Note also, that some calculus concepts and skills are not central to any DE topic. For example in calculus *concepts*, approximation, Riemann sums, continuity, and optimization are absent and in *skills*, polar coordinates and epsilon-delta proofs are also not used. We caution the assumption that because they are not explicitly listed as necessary that they are not utilized anywhere in DEs. Derivations of important equations in DEs provide exemplary reasons for this caution as they are supported through the concepts of approximation, continuity, and optimization.

Figure 1. Decomposition of differential equations topics relative to Calculus Content Framework (Sofronas et al., 2011)

		common topics covered by both textbooks																							
	topics	classification	FOL, variable coeffs	separable eqns	applications with FO	linear vs. nonlinear	autonomous eqns	popluation dynamics	exact eqns	integrating factor	existence / uniqueness	SO linear	fundamental solns	wronskian	complex/ repeated roots	undetermined coeffs	applications (vibrations)	higher order eqn	two-point BVP's	PDE's	Fourier series	separation of variables	heat, wave, laplace eqns	phase planes	systems of linear eqns
calculus concepts	derivative				•			•	•			•				•	•			•		•	•	•	•
	integral			•	•				•	•		•		•		•					•				•
	fundamental thm			•						•															
	limit																•				•		•	•	
	approximation																								
	sequences & series																			•	•		•		
	riemann sums																								
	paramteric & polar eqns																								•
	continuity																								
	optimization																								
calculus skills	derivative computations				•		•	•				•				•	•		•	•	•		•	•	
	integration techniques		•	•	•		•	•	•	•		•		•			•		•	•	•		•		
	limit calculations				•			•									•								
	sequences/series																				•	•	•		
	algebraic expressions		•	•	•				•			•	•	•	•	•	•		•	•	•	•	•	•	
	area/volume				•																				
	parametric eqns																								•
	polar coord.																								
	trig manipulations		•	•	•		•		•			•			•		•		•	•		•	•		•
	logarithms and exponentials		•	•	•				•	•		•		•	•	•			•	•			•		•
	epislon-delta																								
	listening & reading comprehension							•			•														
	definitions & notations	•				•					•							•							•

Epistemological Mismatches Through our mathematics-in-use analysis technique, we uncovered epistemological mismatches within these calculus concepts: derivative, integral, fundamental theorem of calculus, limit, sequences and series, parametric and polar equations, and continuity. Table 1 provides a summary of each of these mismatches and then elaboration on each concept follows. We draw attention to the fact that despite some concepts having no visible mismatches, vital concepts, such as derivative, are vastly mismatched.

Derivative Concept Throughout calculus, the concept of derivative is used to produce a numerical value. Students are taught computational methods for taking the derivative of a function and finding the derivative at a point for a given function. In most cases, solutions to derivative exercises are numerical values. In DEs, the derivative is used as an operator, specifically as the inverse to the integral operator.

Integral Concept During calculus courses, the concept of integral appears throughout the course. For example, it is embedded in: the area under a curve, the area between two curves; Riemann sums and summations; improper integrals definite and indefinite anti-derivatives; solids of revolution; work-energy relations; accumulation; measurement of attributes (e.g., arc length, surface area, volume, center of mass); fundamental theorem of calculus. Many times, computation is emphasized over application of the concept of integral. In contrast integrals in DEs are viewed and used mainly as the inverse operation of derivative to produce

Table 1: Comparison of uses of calculus ideas in calculus and differential equations

	Calculus	Differential Equations
derivative	computation; rate of change	rate of change; algebraic object; invertible operation
integral	anti-derivative; measurement of area, volume, accumulation	technique for solving differential equations
fundamental theorem of calculus	formal justification for using anti-derivatives instead of definite integrals; short cut for computing certain derivatives and definite integrals	creates free parameter for satisfying initial/boundary conditions
limit	algebraic computation; exposure to formal (i.e., ϵ-δ) definitions; basis for derivative definition	relates discrete and continuous models of a situation
sequences & series	using theorems to check for convergence; writing common functions in polynomial bases	means for conveniently representing functions
parametric & polar equations	alternative representations of functions	introducing auxiliary equations; tool for reducing order; tool for representing solutions graphically
continuity	property to be checked	mathematical tool for ensuring desirable mathematical properties

a family of functions. They are used also to compute coefficients of Fourier series. The overall epistemological mismatch is that while in calculus, students expect to obtain numerical values when using integrals while in DEs, it is vital as an invertible operator.

Fundamental Theorem of Calculus Concept The fundamental theorem of calculus provides the relationship between the definite integral and anti-differentiation in calculus courses. It also provides a simplification of computation of definite integrals. While the fundamental theorem of calculus is vital to DEs, it is not explicitly visible because differential equations themselves are collections of derivatives in combination with algebraic procedures that form conditions which are satisfied by a family of functions. It can also be hidden because there are very few cases when definite integrals are expressed in the form of the theorem statement. On the contrary, fixing initial or boundary conditions constrain the choice of a particular anti-derivative. Lastly, the fundamental theorem of calculus is hidden when using methods such as undetermined coefficients or separation of variables. When it is visible, it underpins the integral as the inverse operator of the derivative. Part of the reason why this topic is a mismatch might be due to the fact that derivatives and integrals are not seen as operators in calculus.

Limit Concept In calculus, limits are primarily algebraic computations that replace the brief introduction to epsilon-delta proofs. It forms a basis for the definition of derivative. In DEs the limit concept is used to derive differential equations and to discuss long-term behavior and stability. Whereas in calculus, the limit is treated as a computational exercise, in DEs, it is utilized in reasoning through behaviors of functions.

Sequences & Series Concept In calculus, sequences and series are given as topics in a vacuum. Most of the exercises required in calculus textbooks include reading and applying theorems to determine limits or checking for convergence. Occasionally, students are expected to use series to compute approximations of numbers, such as e or $\frac{pi^2}{6}$. MacLauren, Taylor, and power series are introduced to provide (possibly infinite) sets of monomials for approximating values of various functions, such as e^x, at or near a point of interest. However, rather than focusing on the new representation being a way of defining complex functions in terms of simpler functions, students in calculus are required only to compute higher order derivatives to construct the power series.

The emphasis is inverted in DEs. Power series are called upon regularly for their utility in representations of functions. Using a technique where one assumes the solution function can be represented as a convergent power series, the monomial coefficients can be determined through recursion relations. Fourier series construction is another place where series are used to transform ordinary and partial DEs. To summarize, in DEs, series provide new ways to construct and represent functions while in calculus, sequences provide an introduction to partial sums, which are then used to approximate specific values of functions.

Parametric & Polar Equations Concept Parametric and polar equations are used within calculus to provide an introduction to functions in alternate coordinate systems. Generally, exercises request translation between Cartesian, parametric, and polar systems or computing integrals in these systems. Because parametric equations are so closely related to vector-valued functions in DEs, they are used to convert a higher-order differential equation to a system of first order equations. This process allows the study of evolution of these systems numerically or graphically. Parameterizations are also used in dynamical systems approaches to represent solutions in graphs involving t as the input variable and graphs with the unknown function as the variable, requiring complex shifts in reasoning.

Continuity Concept During calculus courses, continuity is treated as a property to be checked using the vertical line test or limit checking. In DEs, continuity is used as a tool for ensuring desired mathematical properties.

Common Themes The previous subsections have provided a glimpse of the epistemological mismatches between calculus and DEs. One major theme across all examples is that in calculus, functions are treated as actions that produce output numbers given input numbers, whereas in DEs, functions must be viewed as objects. This distinction includes derivatives and integrals.

Interpretation and Concluding Remarks Using the CCF decomposition of DEs content into calculus topics, we reveal what calculus topics must be known, but not how they must be known. Our contributions are toward strengthening the CCF in its practical application of relating calculus-dependent topics to calculus coursework as well as utilizing the framework for exploring conceptual coherence across curricula.

Our results provide further evidence that a large amount of DEs requires proficiency in calculus concepts and skills. Adding to the decomposition with the mathematics-in-use technique showed several major concepts where epistemological mismatch occurs.

Ferguson's (2012) work revealed that there are epistemological mismatches between end-of-calculus knowledge and the following courses from other disciplines (Ferguson, 2012). For example, there were mismatches between what teachers of calculus wanted their students to know versus what users of calculus, such as calculus-based physics course instructors, wanted their students to know. Our results confirm her findings, and like Raman's (2002, 2004) findings of epistemological mismatches between pre-calculus, calculus, and analysis, they show conclusively that even within mathematics, instructors need a new approach to aid in mathematical coherence of their courses.

We chose to use DEs as an example setting, but there is a growing body of evidence in the literature that our expectations for calculus knowledge are out of line with our expectations for post-calculus courses, regardless of whether the students are following a mathematics major track or not. Thus, in order to address the epistemological incoherence we must explicitly focus on how we expect our students to know the content that the community decides is important. One way to do this through thorough evaluation of the phenomenology and the mathematics-in-use of the mathematical concepts, objects, tasks, examples, and exposition that we show our students. Our major suggestion would be to encourage coherence along phenomenological arcs from pre-calculus, to calculus, to post calculus, which would require cooperation among mathematics instructors and work to identify the "big ideas" and the concept-eliciting tasks that can codify them. Calculus is a multi-purpose course that is intended to serve many disciplines and to support many topics in mathematics. For it to be a functional course, we need to examine its content from many views, both post- and pre-calculus, and in our opinion, from the perspective of epistemological coherence.

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