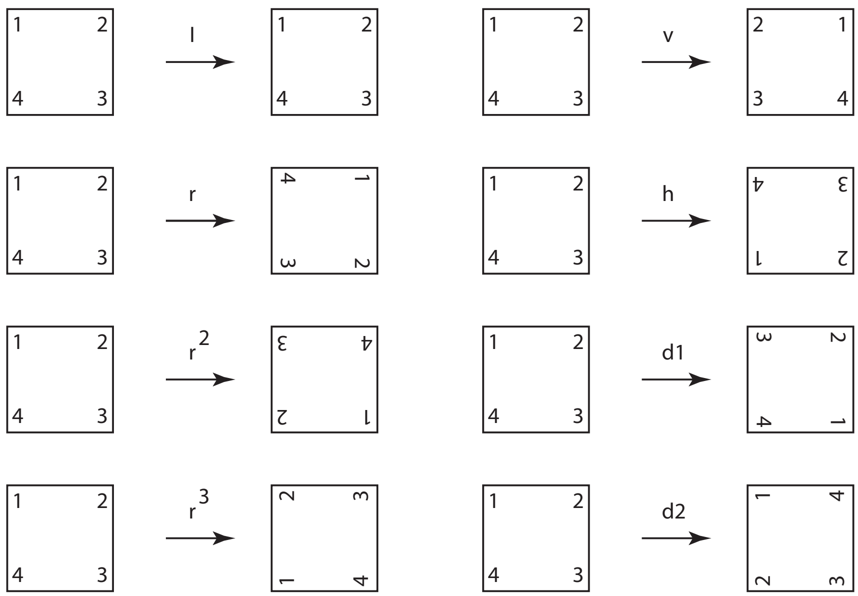
HW 6 Math 5

1. The eight symmetries of the square are shown below. The rotation is denoted r and the "flips" are called $v, h, d1$ and $d2$ for "vertical", "horizontal", "diagonal (one)" and "diagonal (two)", respectively.



   (a) Complete the following multiplication table of the symmetry group of the square. This groups is called the *dihedral group of order 8* and is denoted D_4. More generally, the symmetry group of a regular n-gon is called the dihedral group D_n, and has $2n$ elements.

	I	r	r^2	r^2	v	h	d1	d2
I								
r								
r^2								
r^3								
v								
h								
d1								
d2								

   (b) Is the symmetry group of the square commutative? Explain?

2. Continuing with the symmetry group of the square, show that r and v will generate the group. That is, show that every element of the group can be expressed in terms of r and v alone. In fact, show that every element of the group if equal to one of the following: $I, r, r^2, r^3, v, vr, vr^2$ and vr^3. (In general, for any n, the dihedral group D_n can be generated by the rotation and a single flip across one axis of bilateral symmetry.)

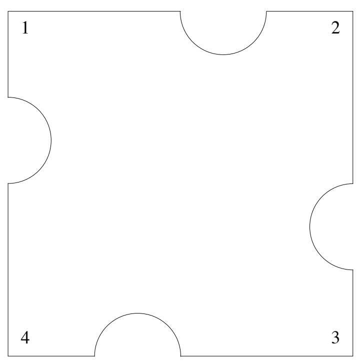
3. Each symmetry of the square can be thought of as a permutation of its four corners. Thus the elements of D_4 can also be thought of as elements of S_4. Explicitly write down this correspondence. That is, for each of the elements $I, r, r^2, r^3, v, vr, vr^2$ and vr^3, write down the corresponding permutation in S_4 using cycle notation. This shows that D_4 is a *subgroup* of S_4. Actually, it shows that D_4 is *isomorphic* to a subgroup of S_4. The elements of D_4 are technically not elements of S_4 (they are symmetries of the square, not permutations of four things) so they cannot be a subgroup of S_4, but instead they *correspond* to eight elements of S_4 which form a subgroup of S_4. This correspondence is called an isomorphism.

4. Using r and v as generators, draw the Cayley graph of D_4. Use two different colors for the edges, one color for r and one color for v. HINT: It will be helpful to use the multiplication table that you made for D_4, and even more helpful if you replace $h, d1$ and $d2$ in that table with their representations as vr, or vr^2, or vr^3.

5. Suppose we destroy some of the symmetry of the square, that is, we replace the square with something that is less symmetric. Here are two possible ways to do this:

   (a) Replace the square with a rectangle that is not a square.

   (b) Take "bites" out of the sides of the square to create a "jigsaw puzzle piece."



   Analyze the symmetry groups of each of these objects. How many elements are in each group? Name the elements using the same names given in Exercise 1. What are nice sets of generators for each group? Draw Cayley graphs for each group using the sets of generators you chose. Each group still corresponds to a set of permutations of the corners. Write down this correspondence. Do the symmetry groups of these two objects correspond to subgroups of D_4? To subgroups of S_4? Do the Cayley graphs of the symmetry groups of these two objects show up nicely inside the Cayley graph of D_4?

6. There are lots of different pairs of permutations in S_4 that will generate the group. One pair of generators is $(1\,2\,3)$ and $(1\,2\,3\,4)$. Using these as generators, draw the Cayley graph of S_4. Use two colors for the edges. You should get a very pretty picture, so rearrange your graph if necessary to make it look good. Does your Cayley graph for D_4 show up inside the Cayley graph for S_4 in a nice way? If not, why not?