Def A group G is <u>generated</u> by a nonempty subset $S \subset G$ if every element of $g \in G$ can be expressed as

$$g = w_1 w_2 \cdots w_n \quad \text{for some } n \in \mathbb{N}$$

and where $w_i \in S$ or $w_i^{-1} \in S$ for all i. We say $w_1 w_2 \cdots w_n$ is a <u>word</u> in S of <u>length</u> n.

Example S_3 is generated by (12) & (123) because every permutation in S_3 can be written as a word in $S = \{(12), (123)\}$.

If $s \in S$, then $1 = ss^{-1}$, so the identity is a word in S. <u>But</u> it is also convenient to allow the "empty word" of length 0 and regard it as the identity.

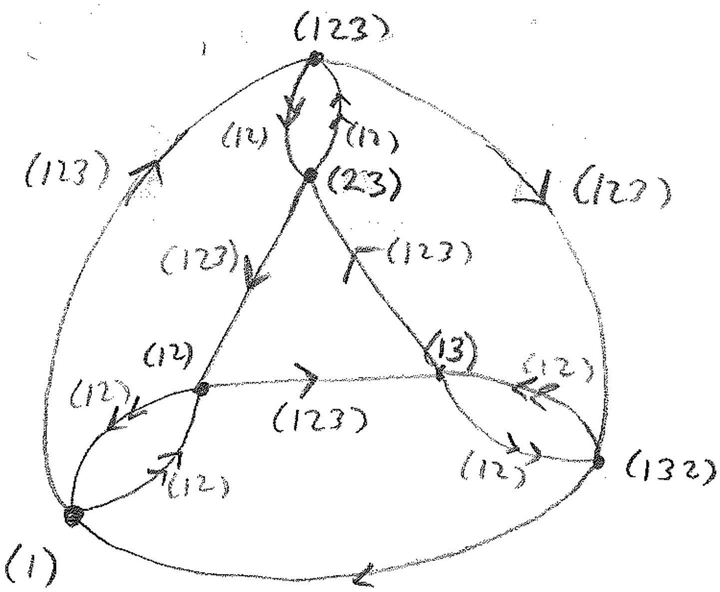
Thm Every group has a set of generators.

Pf We can always take S to be all of G. ∎

If G is a group generated by S,
the <u>Cayley graph of G with respect to S</u>
is the oriented, labeled, graph Γ $\ni$

1) the vertices of Γ are the elements of G, &
2) if $s \in S$ and $g_1 s = g_2$ then there is an oriented edge of Γ going *from* g_1 *to* g_2 & labeled s.

Ex The graph of S_3 with respect to $S = \{(12), (123)\}$ is



Thm Every group has a Cayley graph w.r.t. some set of generators.

Relations and Sets of Defining Relations

Let G be a group generated by S.

A relation in G is an equation $w_1 = w_2$ which is true in G where w_1 and w_2 are words in S.

Notice that if $w_1 = w_2$ then $w_1 w_2^{-1} = 1$ is also a relation. So every relation can be written as $w = 1$ for some word w in S. In this case, the word w is called a relator.

Def If $w = w_1 w_2 \cdots w_n$ is a word in S, a subword is a word $w_i w_{i+1} \cdots w_{i+j}$ where $1 \le i \le n$ & $0 \le j \le n-i$.

Def A word w in S is reduced if ss^{-1} and $s^{-1}s$ are not subwords of w for all $s \in S$.

<u>Ex</u> S_4 is generated by

$S = \{(12), (23), (34)\}$.

$(12)(23)(12)(34)$ is reduced

$(12)\underbrace{(23)(23)}_{ss^{-1}}(34)(12)$ is not reduced

<u>Def</u> Let G be group generated by S and $R = \{w_\alpha = 1\}_{\alpha \in A}$ a set of relations. We say the relation $w = 1$ is a consequence of R if w can be taken to the empty word in a finite sequence of these steps:

1) change w by the introduction or deletion of the subword ss^{-1} or $s^{-1}s$ where $s \in S$

2) change w by the introduction or deletion of the subword w_α where $w_\alpha = 1$ is in R.

A relator w is called <u>trivial</u> if w can be taken to the empty word by only using Step 1).

Example

5

Let G be a group generated by $S = \{a, b\}$.

Suppose $R = \{a^2 = 1, b^3 = 1\}$

the relation $b a^2 b^2 = 1$ is a consequence of R since

$$\underset{\text{delete } a^2}{b a^2 b^2 \longrightarrow} \underset{\text{delete } b^3}{b b^2 = b^3 \longrightarrow} \phi$$

<u>Def</u> Let G be group generated by S and R a set of relations. The set R is called a <u>defining set of relations</u> if every relation is a consequence of R.

<u>Thm</u> Every group G generated by S has a defining set of relations R.

<u>Pf</u>: We will give an algorithm to find R given G and S.

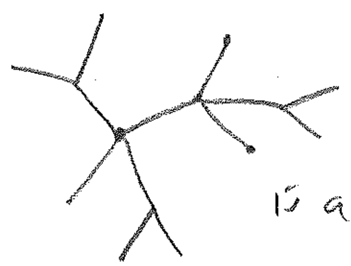
As we explain the algorithm we will give an example to illustrate its use.

<u>Step 1</u> find a Cayley graph Γ for G with respect to S.

<u>Example</u> if $G = S_3$ & $S = \{(12), (123)\}$ then Γ was shown earlier.

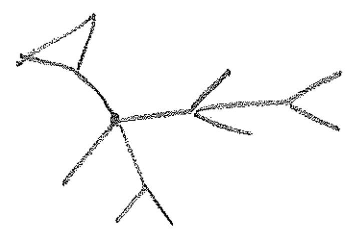
<u>Step 2</u> <u>Choose</u> a maximal tree T in Γ.

<u>Def</u> A <u>tree</u> is a connected graph that <u>contains no loops</u>



is a tree

but



is not a tree

A <u>maximal tree</u> in a graph Γ is a subgraph of Γ that

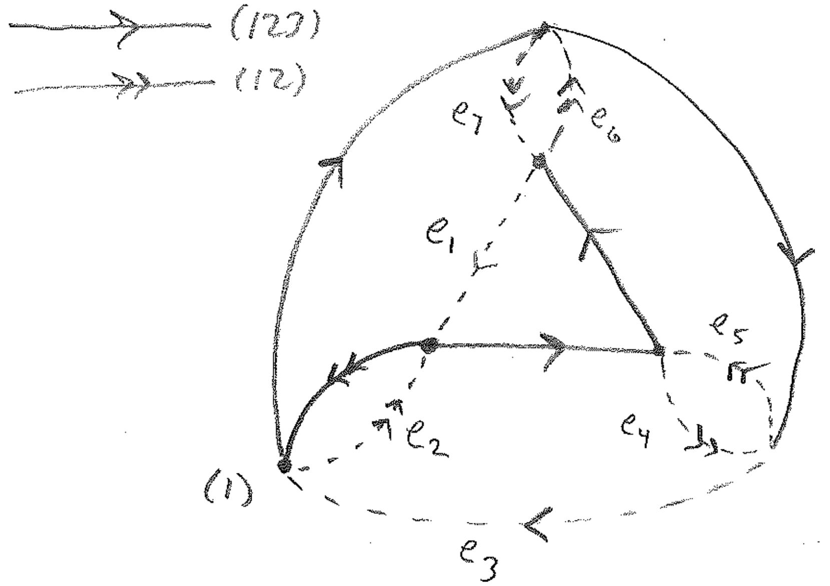
1) is a tree, &

2) contains all vertices of Γ.

Prop Every graph contains a maximal tree.

Pf postponed.

For S_3 we can chose T as shown below



T consists of solid edges, dashed edges are not in T.

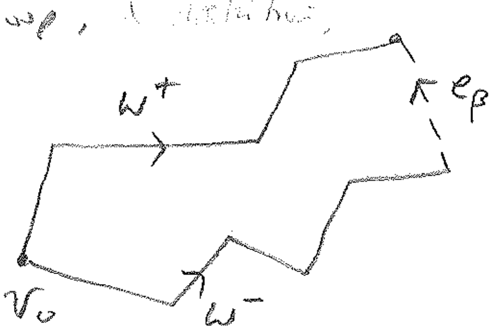
Let $\{e_\beta\}_{\beta \in B}$ be the set of all edges in Γ that are not in T.

Each edge, e_β, gives a relation as follows.
Let v_β^- & v_β^+ be the beginning & ending vertices of e_β. Let v_0 be some vertex of Γ. It's nice to choose 1 for v_0

Because T is a tree there exists a unique path $w^- = w_1 w_2 \cdots w_k$ in T from v_0 to v_β^- and a unique path $w^+ = u_1 u_2 \cdots u_j$ in T from v_0 to v_β^+. We can think of each w_i or u_i as an edge of T and simultaneously as an element of S, or an element whose inverse is in S.

We can now form the loop, [illegible]

$$w^- e_\beta (w^+)^{-1}$$



and corresponding relation $w^- e_\beta (w^+)^{-1} = 1$

Let $w_\beta = w^- e_\beta (w^+)^{-1}$.

In our example there are seven edges in Γ that are not in T.

For each one we list the associated relation.

e_1 $(12)^{-1}(123)^3(12) = 1$

e_2 $(12)^2 = 1$

e_3 $(123)^3 = 1$

e_4 $(12)^{-1}(123)(12)(123)^{-2} = 1$

e_5 $(123)^2(12)(123)^{-1}(12) = 1$

e_6 $(12)^{-1}(123)^2(12)(123)^{-1} = 1$

e_7 $(123)(12)(123)^{-2}(12) = 1$

<u>Claim</u> Every relation is a consequence of $R = \{w_\beta = 1\}_{\beta \in B}$.

<u>Pf</u> Suppose $w = 1$ is any relation. Then starting at $v_0 = 1$ in the Cayley graph and tracing the path w gives a loop. We will show that w is a consequence of R by induction on the number of edges e_β that are contained in w.

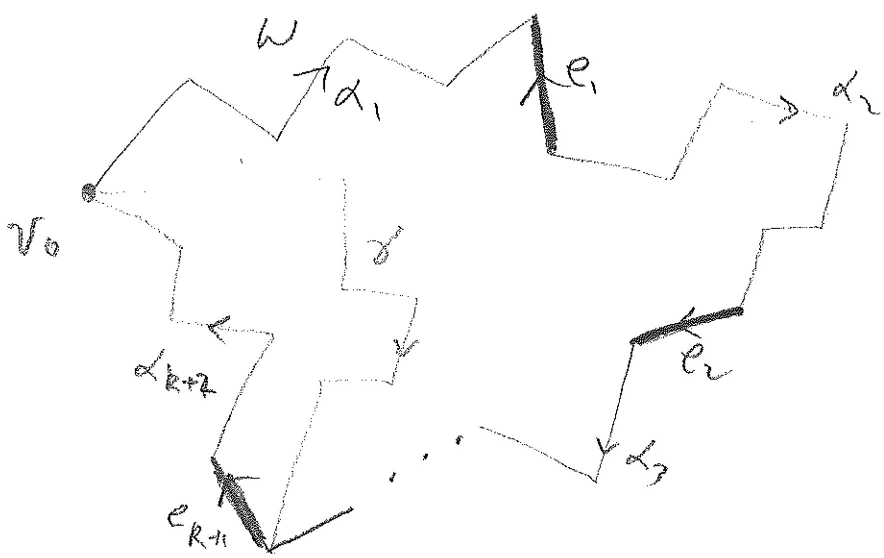
<u>Initial Case</u> w contains no edges e_β.

In this case w lies entirely in the tree T.

Because T is a tree, this means that w can be taken to the empty word by repeated deletion of the subwords ss^{-1} or $s^{-1}s$. (this can be proven by induction on the length of w.)

<u>Induction hypothesis</u> Assume that if w contains k edges from $\{e_\beta\}_{\beta\in B}$ then $w=I$ is a consequence of R.

<u>Induction Step</u> Suppose w contains $k+1$ edges from Γ that are not in T. Call them $e_1, e_2, \dots, e_{k+1}$



We can write w as

$$w = \alpha_1 e_1^{\varepsilon_1} \alpha_2 e_2^{\varepsilon_2} \cdots \alpha_{k+1} e_{k+1}^{\varepsilon_{k+1}} \alpha_{k+2}$$

where each α_i is a path in T
and each $\varepsilon_i \in \{1,-1\}$ depending on whether
we traverse e_i in the direction of its orientation
or not.

Now let γ be a path in T from v_0 to
the end of α_{k+1}, which is a vertex of e_{k+1}.

Change w by using step 1 repeatedly to
insert $\gamma^{-1}\gamma$:

$$w = \underbrace{\left(\alpha_1 e_1^{\varepsilon_1} \alpha_2 e_2^{\varepsilon_2} \cdots \alpha_{k+1} \gamma^{-1}\right)}_{w'} \left(\gamma \varepsilon_{k+1} \alpha_{k+2}\right)$$

Now w' is a loop that contains k edges from
Γ not in T. So we are assuming ...
w' is a consequence of R. Hence we may
take w' to the empty word by steps 1 & 2).
We now have

$$w \longrightarrow w'(\gamma \varepsilon_{k+1} \alpha_{k+2}) \longrightarrow \gamma \varepsilon_{k+1} \alpha_{k+2} \longrightarrow \phi$$

the last step follows because $\gamma \varepsilon_{k+1} \alpha_{k+2}$ is
a loop containing exactly one edge of Γ not in T.

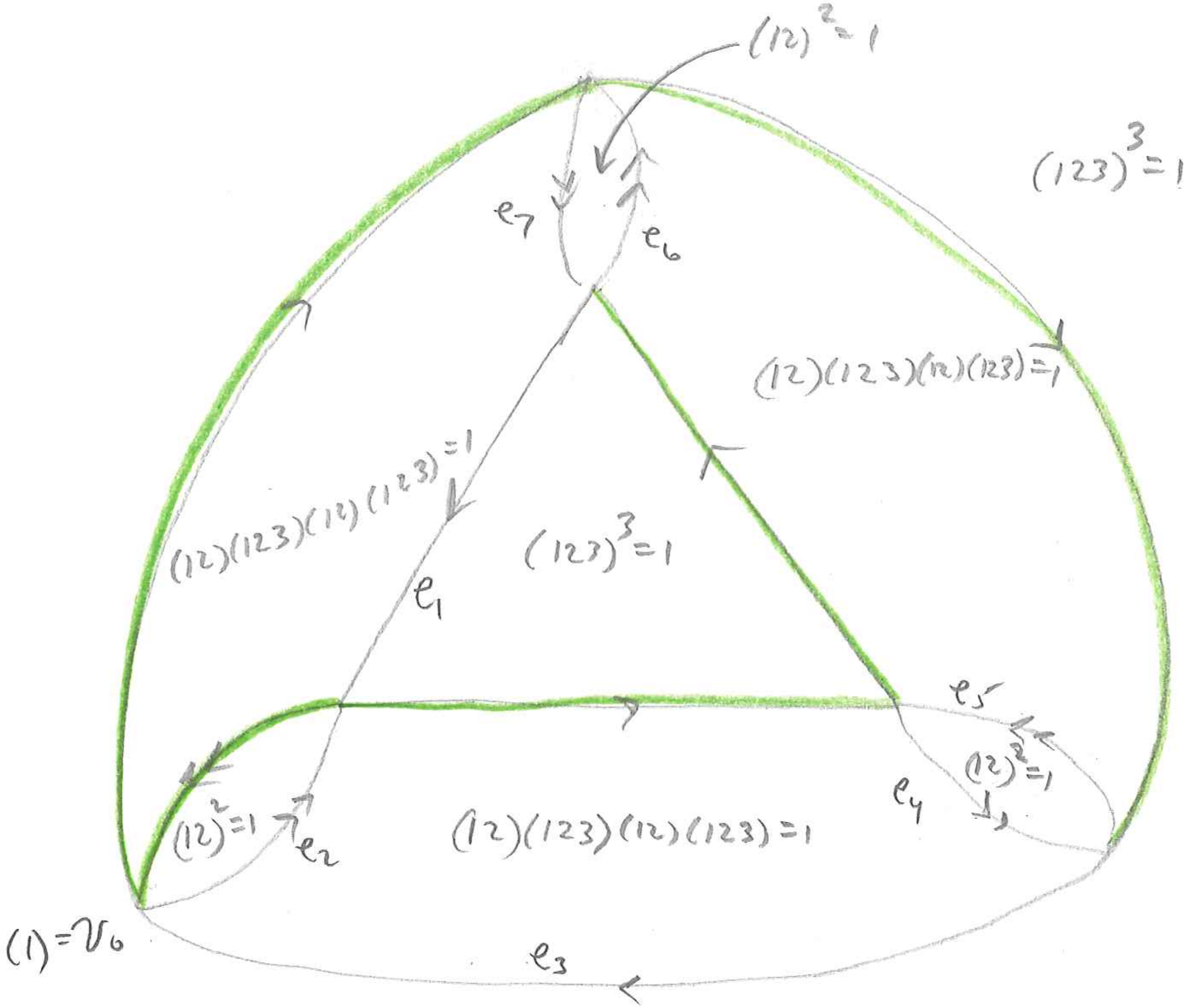
So in our example we obtain a set of seven defining relations. But this set can be further reduced to a set of three defining relations using the following:

Theorem Suppose Γ is planar & G is finite* For each face f of Γ (a complementary region of $\mathbb{R}^2 - \Gamma$) let $w_f = 1$ be the relation given by the perimeter of f. Then the set of relations $\{w_f = 1\}_{f \in \text{Faces of } \Gamma}$ is a defining set of relations.

* We don't need G finite. But we need that any simple closed path in Γ bounds a finite # of faces on one side.

In our example the faces give the following relations:

Explained later.



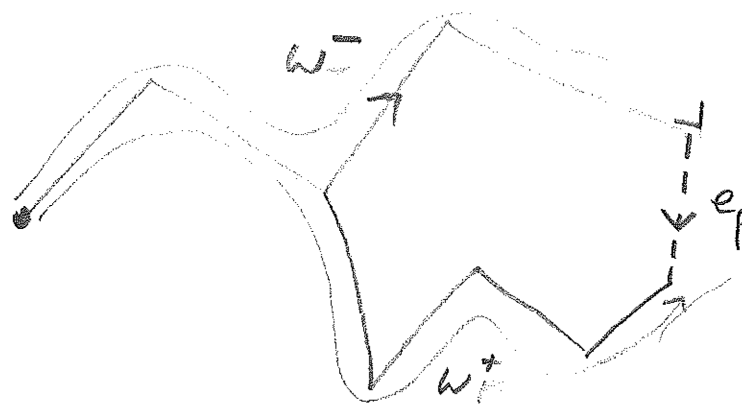
There are 8 faces and we get the "face" relations

$(123)^3=1$ (twice)

$(12)^2=1$ (three times)

$(12)(123)(12)(123)=1$ (three times)

Proof of Theorem It suffices to show that each edge relation $w^- e_\beta (w^+)^{-1} = 1$ is a consequence of the face relations. Consider $w^- e_\beta (w^+)^{-1}$

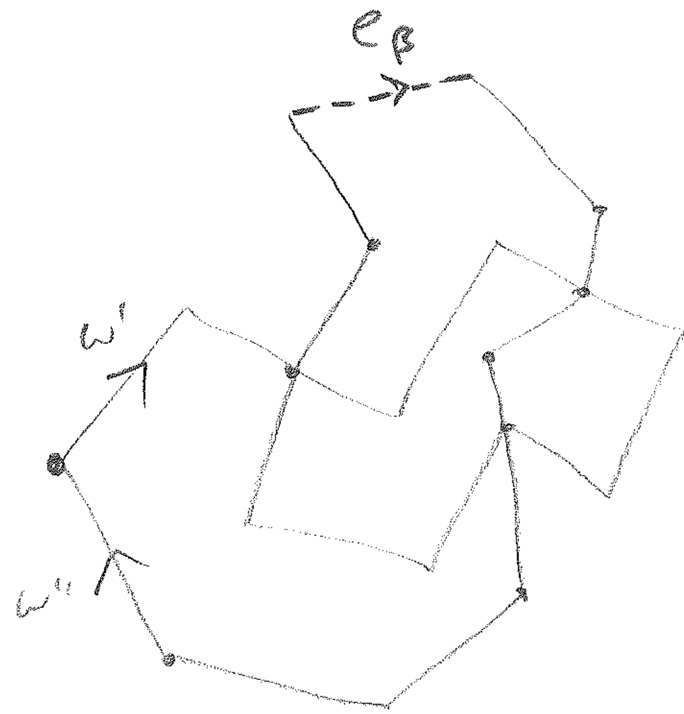


it might be that the beginning of w^- is the end of w^+.

Now $w = \alpha w' e_\beta w'' \alpha^{-1}$

Now $w' e_\beta w''$ is a loop that traverses no edge twice. If $w' e_\beta w'' \longrightarrow \phi$ by 1) & 2) then $w = \alpha w' e_\beta w'' \alpha^{-1} \longrightarrow \alpha\alpha^{-1} \longrightarrow \phi$.

So focus on $w' e_\beta w''$. This loop is a simple closed curve in the plane because w' & w'' are in T. (we can't have

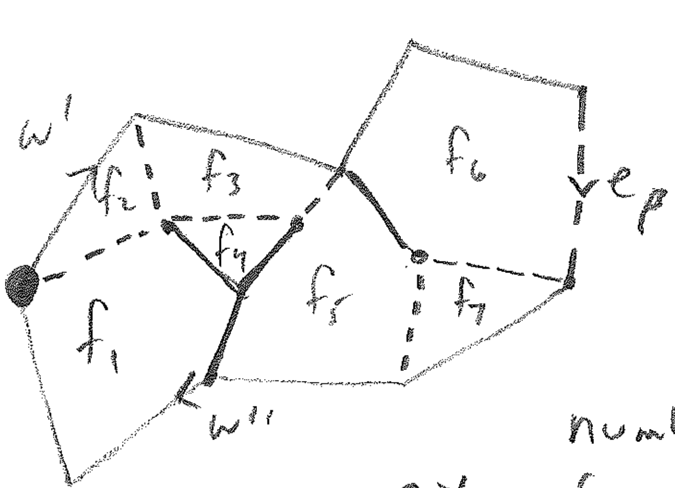


If the loop crosses itself then $\exists$ loops in T. Contradiction.

Now we'll use the <u>Jordan Curve Theorem:</u>

Every simple closed curve in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into two regions, an inside & an outside.

So in our case $w' e_\beta w''$ has an inside region which must be a union of faces.



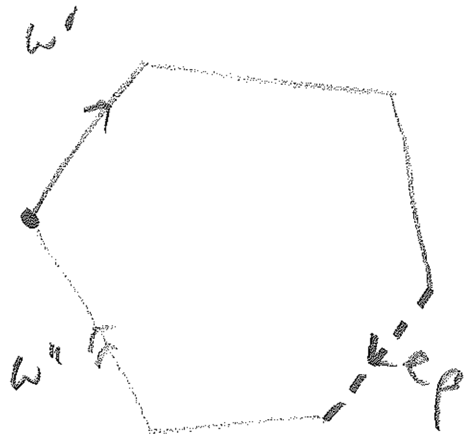
Solid edges in T
dashed edges not in T.

We're assuming that <u>there are only a finite</u> number of faces on one side of $w' e_\beta w''$.

Call them $f_1, f_2, \dots, f_m$

We now induct on m.

If m is one, it appears as



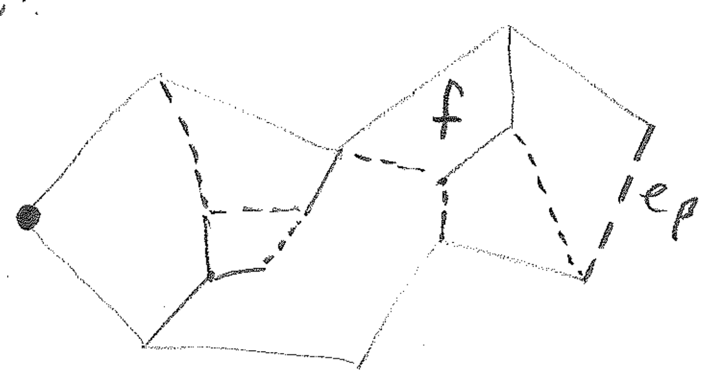
and the word $w'epw'' \longrightarrow \phi$

in one step, since $w'epw''$ is a face relation.

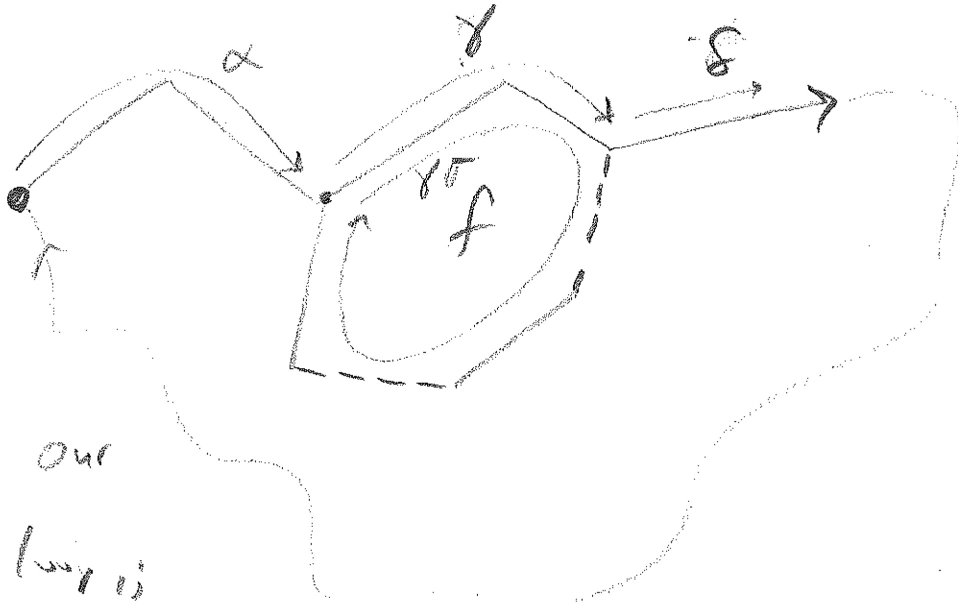
Now suppose the result is true if there are m faces

Assume $w'epw''$ bounds m+1 faces.

There must be a face f containing an edge of $w'epw''$.



We will change $w'\epsilon p w''$ so as to detour around f, thus getting a loop with one less face.



our loop is

$\alpha\gamma\delta$ but the boundary of f is $\gamma\sigma = 1$

Now $\alpha\gamma\delta \xrightarrow[1)]{} \alpha\gamma\sigma\sigma^{-1}\delta$, insert $\sigma\sigma^{-1}$

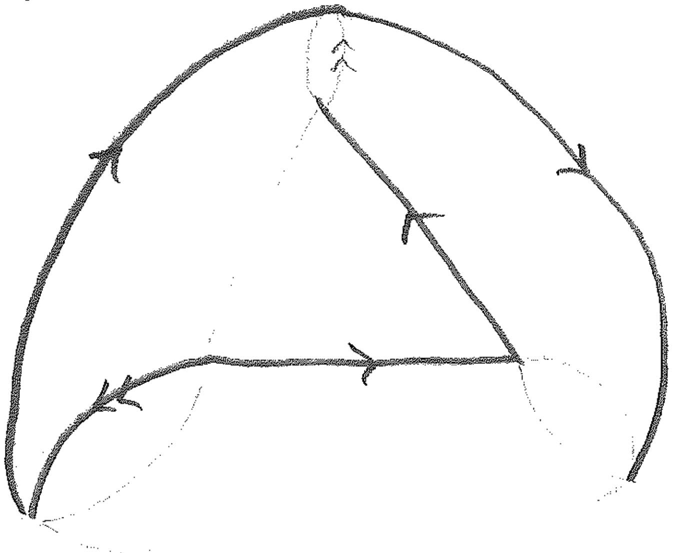
$\longrightarrow \alpha\sigma^{-1}\delta$ delete $\gamma\sigma$ face relator

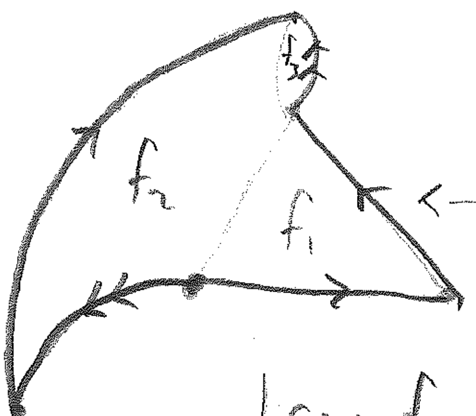
the loop now has one less face inside

So by inductive hypothesis can be taken to ϕ by moves 1) & 2) using face relations

2

Example Consider our relation from edge e_6





there are three faces inside

$$(12)^{-1}(123)^{2}(12)(123)^{-1}$$

insert $(123)(123)^{-1}$

$$\downarrow$$

$$(12)^{-1}\cancel{(123)^{3}}(123)^{-1}(12)(123)^{-1}$$

$$\downarrow$$

$$(12)^{-1}(123)^{-1}(12)(123)^{-1}$$

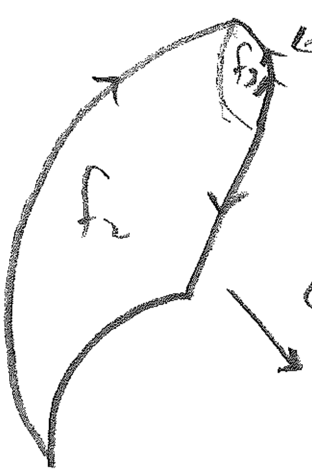
$$\downarrow$$

$$(12)^{-1}(123)^{-1}(12)^{-1}\cancel{(12)(12)}(123)^{-1}$$

$$\downarrow$$

$$(12)^{-1}(123)^{-1}(12)^{-1}(123)^{-1}$$

cross f_1



cross face f_3

cross f_2

ϕ

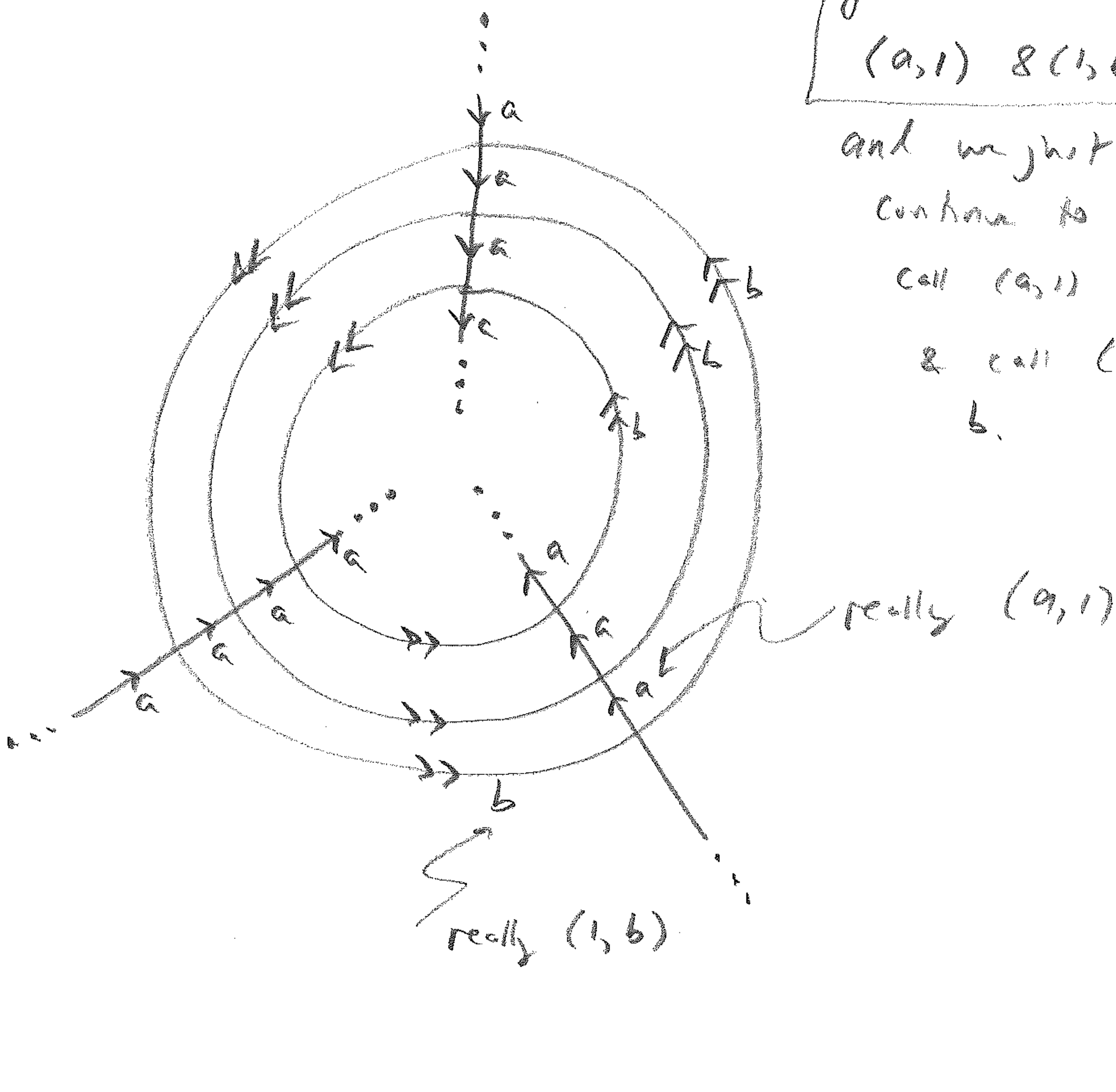
Why Planar is not enough.

Consider $C_\infty \times C_3$ Say C_∞ is generated by a & C_3 generated by b.

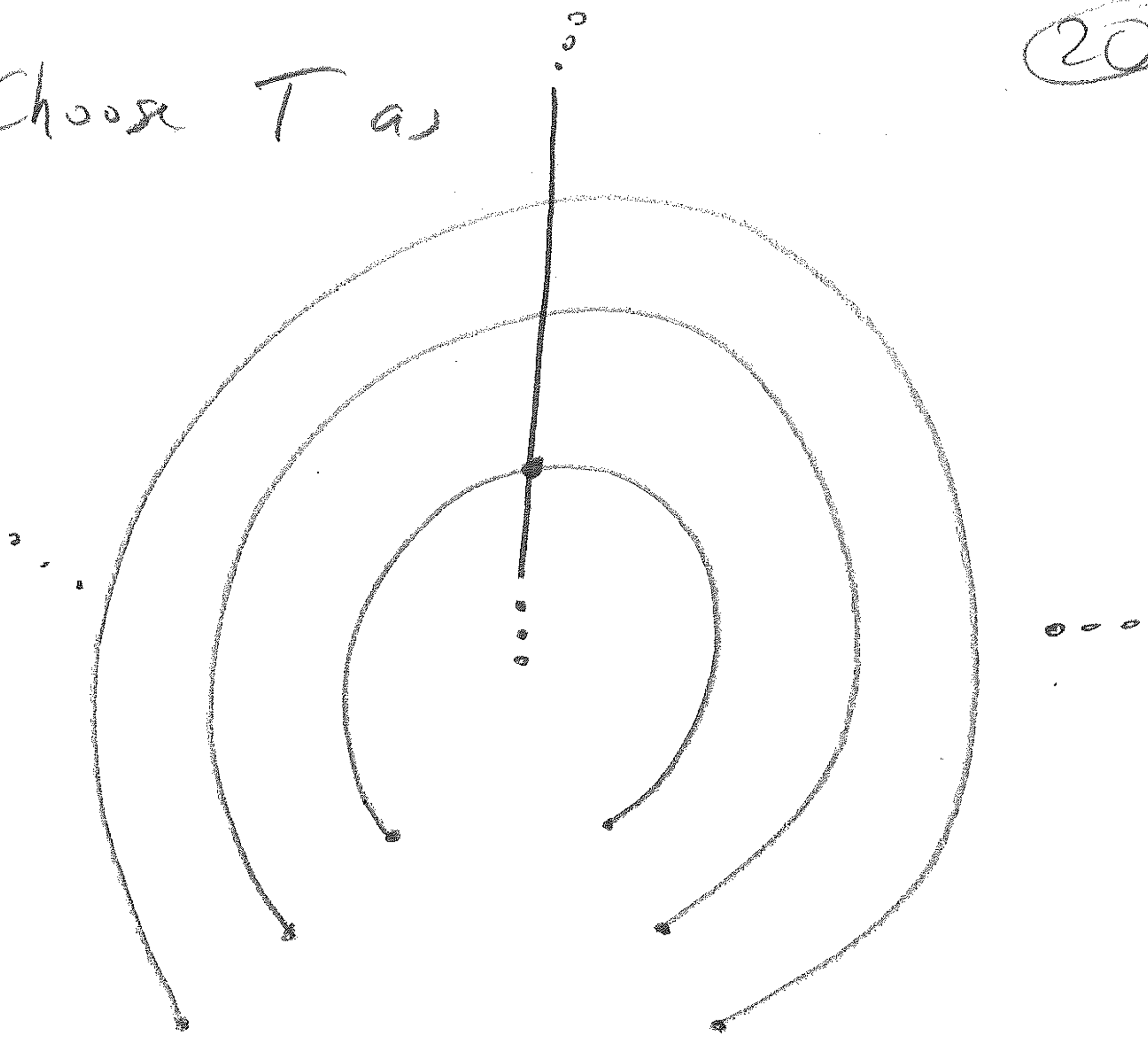
<u>Claim</u> the Cayley Graph is

$\boxed{\text{Generators are } (a,1) \;\&\; (1,b)}$

and we just continue to call $(a,1)$ a & call $(1,b)$ b.



Choose T as



Check that each face relation is $aba^{-1}b^{-1}=1$

But one of the edge relations is $b^3=1$.

We can never derive $b^3=1$ from $aba^{-1}b^{-1}=1$ because as we perform moves 1) & 2) to b^3 the total sum of the exponents will remain 3. inserting ss^{-1} or $aba^{-1}b^{-1}$ does not change exponent sum. Our proof fails because the loop b^3 does not bound a finite # of faces!