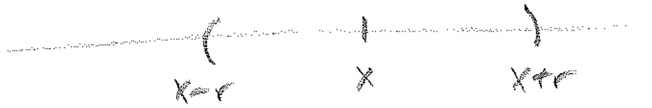
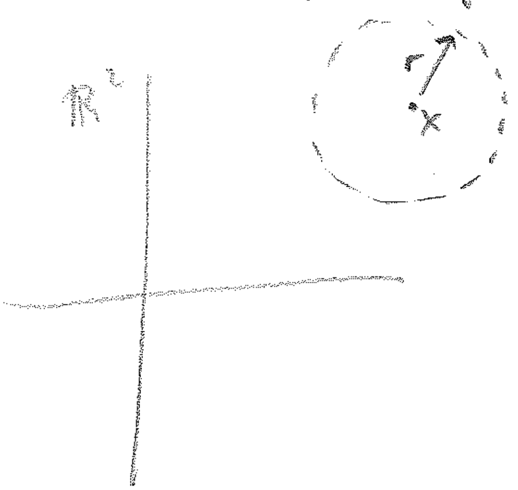
Def An <u>open ball</u> <u>centered at x</u> <u>of radius r</u> <u>in $\mathbb{R}^n$</u> is the set

$$\mathring{B}_x(r) = \{ y \in \mathbb{R}^n \mid |x-y| < r \}.$$

~~Note~~ Notice that in $\mathbb{R}^1$, an open ball is an open interval $(x-r, x+r)$



In $\mathbb{R}^2$, an open ball is an open disk



in $\mathbb{R}^3$ it's really a "ball".

Def Suppose $X \subset \mathbb{R}^n$ is a subset.

An <u>open cover</u> of X is a collection of open balls in $\mathbb{R}^n$ whose union contains X.

A <u>subcover</u> of a cover is a subcollection of the cover.

Thm (Heine-Borel Theorem) Suppose $\mathcal{G}$ is an open cover of $I = [0,1] \subset \mathbb{R}^1$. Then there exists a finite subcover of $\mathcal{G}$ that is also an open cover of I.

Pf Suppose not. Then not both $[0, \frac{1}{2}]$ & $[\frac{1}{2}, 1]$ are contained in a finite subcover of $\mathcal{G}$. (If both were, then $[0,1]$ would be.)

Let I_1 be either $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$ $\ni$ I_1 is not contained in a finite subcover of $\mathcal{G}$.

Now split I_1 in half. One of its halves is not in a finite subcover of $\mathcal{G}$. Let that half be I_2. Continuing in this way we obtain a sequence

$$I \supset I_1 \supset I_2 \supset I_3 \supset \cdots$$

$\ni$ $\forall k$, I_k is not contained in a finite subcover of $\mathcal{G}$.

Claim $\bigcap_{k=1}^{\infty} I_k$ is a single point P.

Pf ~~the left endpoints of the intervals,~~

Suppose $I_k = [l_k, r_k]$

Given k, we have $I_n \subset I_k$ for all $n \geq k$.

Hence $l_k \leq l_n \quad \forall\ n \geq k$

$l_k < r_n \quad \forall\ n \geq k$

$l_n < r_k \quad \forall\ n \geq k$

& $r_n \leq r_k \quad \forall\ n \geq k$.

So the sequence $\{l_i\}_{i=1}^{\infty}$ forms an increasing sequence bounded above by 1.

Hence it must converge to a limit $L \leq 1$ (this is a deep property of the real numbers)

Also, the sequence $\{r_i\}_{i=1}^{\infty}$ is a descending sequence bounded below by 0

so must converge to some limit $0 \leq R$.

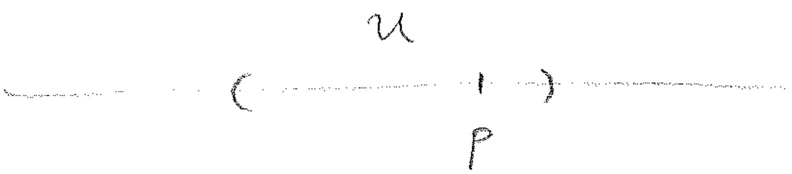
We cannot have $R < L$. If so then $\exists$ n with $r_n < l_n$.

We cannot have $L < R$. If so then $|l_n - r_n| \geq R - L > 0 \quad \forall n$. But $\lim_{n \to \infty} |l_n - r_n| = 0$.

So $R = L$. Let $P = R = L$.

Since $\ell_n \le p \ \forall n$ & $p \le r_n \ \forall n$, we have $p \in I_n$ for all n. Hence $p \in \bigcap_{n=1}^{\infty} I_n$. (In fact $p = \bigcap_{n=1}^{\infty} I_n$, but we will only need that $p \in I_n \ \forall n$.)

Now $\mathcal{C}$ is an open cover. So p is in some open interval $\mathcal{U} \in \mathcal{C}$. Hence



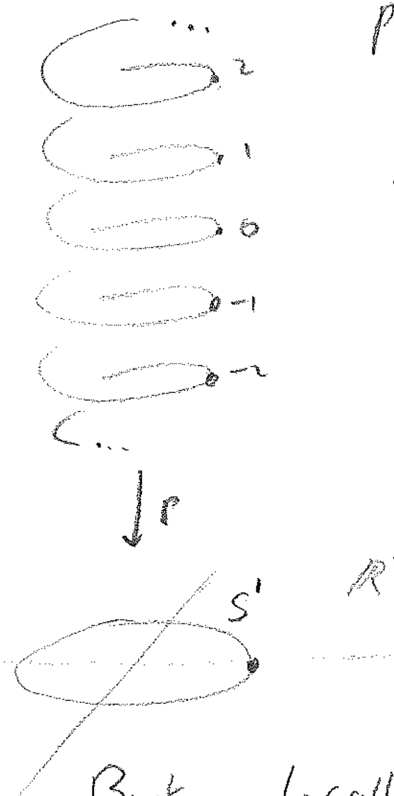
$\exists \ n \ni p \in I_n \subset \mathcal{U}$, because $\lim_{n\to\infty} |\ell_n - r_n| = 0$. But this is a contradiction, because we were assuming that $\forall n$, I_n was not in a finite subcover of $\mathcal{C}$. $\blacksquare$

Covering Spaces

Let $p: \mathbb{R}^1 \longrightarrow S^1$ be defined as

$p(x) = (\cos 2\pi x, \sin 2\pi x) \in S^1 \subset \mathbb{R}^2$

It's nice to picture this as the projection of the helix to the circle.



The map p takes infinitely many points of R to any given point of S^1.

So p is <u>NOT</u> a bijection & hence has no inverse.

But <u>locally</u> p has an inverse.

By this I mean, if we <u>restrict</u> p to a small enough interval of $\mathbb{R}$, then that restricted function has an inverse. In particular, consider the open interval $(k, k+1)$ when k is any integer.

p takes $(k, k+1)$ bijectively to the set

$\mathcal{U} = S^1 - \{(1,0)\}$. And we can define

an inverse $f : \mathcal{U} \longrightarrow (k, k+1)$ as

$$f\left(\left(\cos\theta, \sin\theta\right)\right) = k + \frac{\theta}{2\pi}.$$

~~Now~~ here $\theta \in (0, 2\pi)$ since $(1,0) \notin \mathcal{U}$.

Now $(p \circ f)((\cos\theta, \sin\theta)) = p\left(k + \frac{\theta}{2\pi}\right)$

$$= \left(\cos\left(2\pi\left(k + \frac{\theta}{2\pi}\right)\right), \sin\left(2\pi\left(k + \frac{\theta}{2\pi}\right)\right)\right)$$

$$= (\cos\theta, \sin\theta)$$

So $p \circ f = id$

& Suppose $x \in (k, k+1)$. Then

$x = k + x'$ with $x' \in (0,1)$

& $(f \circ p)(x) = f(\cos 2\pi x, \sin 2\pi x)$

$= f(\cos 2\pi x', \sin 2\pi x')$

$= k + \dfrac{2\pi x'}{2\pi} = k + x' = x.$

So $f \circ p = id$

Similarly, If we restrict p to the interval $(k-\tfrac{1}{2}, k+\tfrac{1}{2})$, $k \in \mathbb{Z}$

then p takes $(k-\tfrac{1}{2}, k+\tfrac{1}{2})$ bijectively to

$V = S^1 - \{(-1,0)\}$ & has an inverse

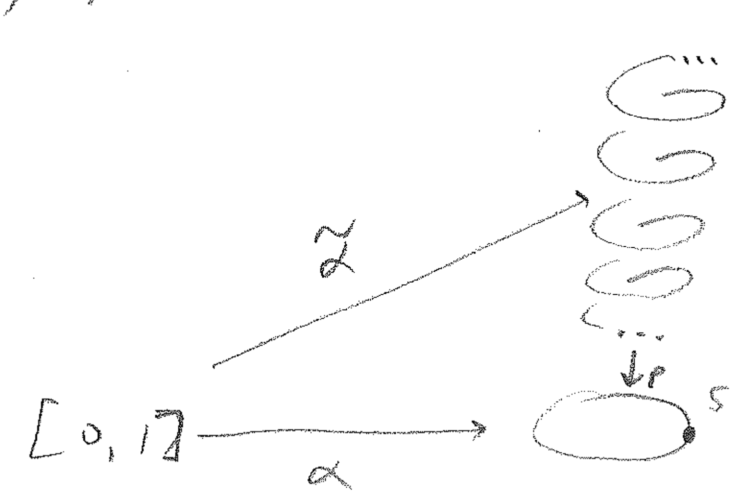
$$g : V \longrightarrow (k-\tfrac{1}{2}, k+\tfrac{1}{2}).$$

This is an <u>example</u> of a covering space.

<u>Def</u>: Let $\alpha : [0,1] \longrightarrow S^1$ be a loop based at $(1,0)$. A <u>lift</u> of α is

a path $\tilde{\alpha} : [0,1] \longrightarrow \mathbb{R}^1$ $\ni$

$$(p \circ \tilde{\alpha})(t) = \alpha(t) \quad \forall \; t \in [0,1]$$



Note that we must have $\tilde{\alpha}(0) \in \mathbb{Z}$ & $\tilde{\alpha}(1) \in \mathbb{Z}$

<u>Lemma</u> (Path-Lifting Lemma)

Suppose $\alpha : [0,1) \longrightarrow S^1$ is a loop based at $(1,0)$. Given any $n \in \mathbb{Z}$ $\exists$ a lift

$\tilde{\alpha} : [0,1) \longrightarrow S^1$ $\ni$ $\tilde{\alpha}(0) = n$ &

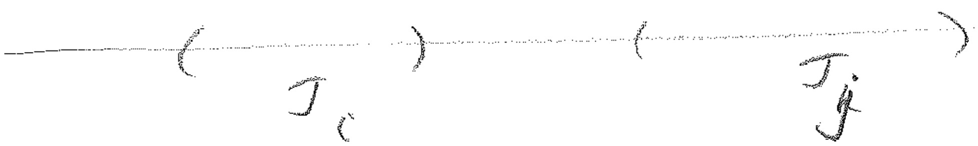
$(p \circ \tilde{\alpha})(t) = \alpha(t)$ $\forall t$.

<u>Pf</u>: Because α is continuous, for each point $t \in [0,1]$ $\exists$ $\delta > 0$ $\ni \forall x$ with $x \in [0,1]$ & $|x - t| < \delta \implies \alpha(x) \in \mathcal{U} = S^1 - \{(1,0)\}$

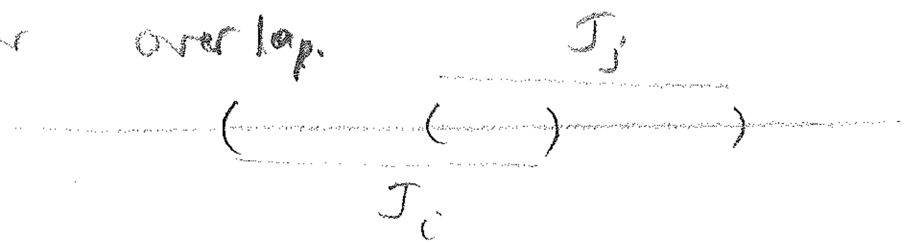
<u>OR</u> $\ni \forall x$, $x \in [0,1]$ & $|x - t| < \delta \implies$ $\alpha(x) \in V = S^1 - \{(-1,0)\}$.

So, containing each point $t \in [0,1]$ we have produced an open interval. Thus the collection of these open intervals provides an open cover $\mathcal{C}$ of $[0,1]$. By the Heine-Borel theorem, $\exists$ a finite subcover of $\mathcal{C}$, call these open intervals $J_1, J_2, \ldots, J_k$.

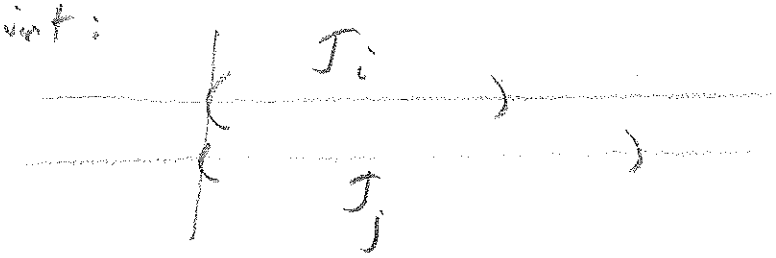
We may assume $J_i \not\subset J_j$ for any pair i,j. If so, we could throw away J_i and still have a finite open cover of I by open intervals. Hence for each pair $i \neq j$, J_i & J_j either are disjoint



or overlap.



No two J_i have the same left (or right) end point:



same left endpoint $\Rightarrow$ one of the intervals is contained in the other

So, we can reorder the intervals

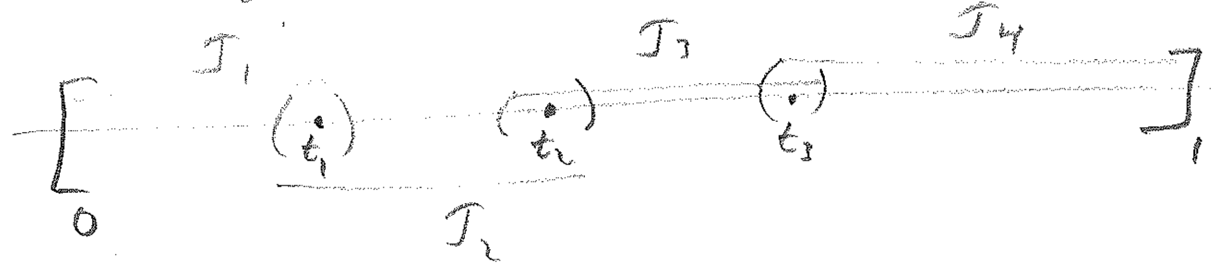
$J_1, J_2, .., J_k$ ∋ the left endpoints

form ~~an~~ a strictly increasing sequence.

We must now have $J_i \cap J_{i+1} \neq \emptyset$.

Hence we can pick points $0 = t_0 < t_1 < t_2 < \cdots t_{k-1} < t_k = 1$

∋ $t_i \in J_i \cap J_{i+1}$ for each i



Now $[t_i, t_{i+1}] \subset J_{i+1}$.

We are now going to construct $\tilde{\alpha}(t)$.

$\alpha(0) = (1,0)$ so $\alpha([0,t_1]) \subset V$.

We're given $n \in \mathbb{Z}$ & want $\tilde{\alpha}(0) = n$.

restricting p to $(n-\frac{1}{2}, n+\frac{1}{2})$ we <u>define</u> $\tilde{\alpha}$

on $[0,t_1]$ as $\tilde{\alpha}(t) = g\,\alpha(t)$

where $g : V \longrightarrow (n-\frac{1}{2}, n+\frac{1}{2})$ is the inverse of

p restricted to $(n-\frac{1}{2}, n+\frac{1}{2})$.

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Now $\tilde{\alpha}(t_1) \in (n-\frac{1}{2}, n+\frac{1}{2})$

and $\alpha([t_1, t_2]))$ is contained in either V or U,

So we define $\tilde{\alpha}$ on $[t_1, t_2]$ by using the

inverse ~~f~~ f or g. ~~α~~

Continuing in this way, we lift α to $\tilde{\alpha}$

on each successive interval $[t_i, t_{i+1}]$

until we have lifted all of α. ∎

Lemma (Uniqueness of lifts) Let $p: \mathbb{R}^1 \to S^1$ be the covering map

$p(x) = (\cos 2\pi x, \sin 2\pi x)$.

Let $\alpha : [0,1] \to S^1$ be a loop based at $(1,0) \in S^1 \subset \mathbb{R}^2$. Suppose

$\tilde{\alpha} : [0,1] \to \mathbb{R}$ is a path $\ni$

i) $\tilde{\alpha}$ is a lift of α, i.e. $p(\tilde{\alpha}(t)) = \alpha(t)$ for all t, and

ii) $\tilde{\alpha}(0) = 0$.

Then $\tilde{\alpha}$ is unique.

pf: Suppose $\tilde{\alpha}$ & $\tilde{\beta}$ are two such lifts.

Let $A = \{ t \in [0,1] \mid \tilde{\alpha}(t) = \tilde{\beta}(t) \}$ &

$D = \{ t \in [0,1] \mid \tilde{\alpha}(t) \neq \tilde{\beta}(t) \}$.

(A for "agree", D for "disagree")

Note that $0 \in A$ so $A \neq \phi$.

Claim 1: Given any point $t \in A$, $\exists\ \delta > 0 \ni$ $x \in [0,1]$ and $|t - x| < \delta \Rightarrow x \in A$.

Similarly,

Claim 2 Given any point $t \in D$, $\exists\ \delta > 0 \ni$ $x \in [0,1]$ and $|t - x| < \delta \Rightarrow x \in D$.

We'll prove claim 1. The proof of claim 2 is similar.

pf of claim 4 : Suppose $t \in A$.

Case I suppose $\alpha(t) \neq (1,0)$. Then $\tilde{\alpha}(t) \notin \mathbb{Z}$ since $p^{-1}(1,0) = \mathbb{Z}$. So $\exists\ k \in \mathbb{Z} \ni k < \tilde{\alpha}(t) < k+1$.

Because $\tilde{\alpha}$ & $\tilde{\beta}$ are continuous, $\exists\ \delta > 0 \ni$ $x \in [0,1]$ & $|x-t| < \delta \Rightarrow k < \tilde{\alpha}(x) < k+1$ and $k < \tilde{\beta}(x) < k+1$. Now if we restrict p to $(k,k+1)$, then p has an inverse, call it $f : S^1 - \{(1,0)\} \longrightarrow (k,k+1)$ given by

$$f((\cos\theta, \sin\theta)) = k + \frac{\theta}{2\pi}$$

Now $p(\tilde{\alpha}(x)) = \alpha(x)$ so $\tilde{\alpha}(x) = f(p(\tilde{\alpha}(x))) = f\alpha(x)$

& $p(\tilde{\beta}(x)) = \alpha(x)$ so $\tilde{\beta}(x) = f(p(\tilde{\beta}(x))) = f\alpha(x)$.

Hence $\tilde{\alpha}(x) = \tilde{\beta}(x)$ for all $x \in [0,1]$ with $|x-t| < \delta$.

From Claim 1 & Claim 2, we get for each point $t \in [0,1]$ an open interval centered at t that is ~~either~~ entirely contained in A if $t \in A$, or entirely contained in D if $t \in D$.

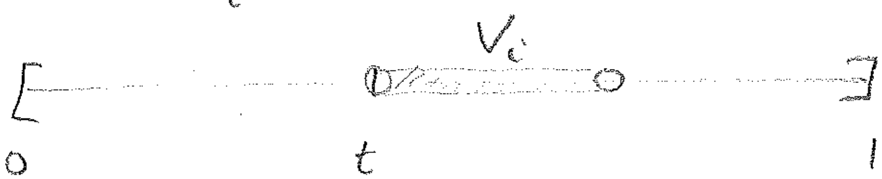
This produces a cover of $[0,1]$ by open intervals. By the Heine-Borel theorem this cover has a finite subcover.

This means we now have a finite collection of open intervals

$U_1, U_2 \dots U_k, V_1, V_2 \dots, V_j$

Such that each $U_i \subset A$, each $V_i \subset D$ and every point of $[0,1]$ is in one of these open intervals. We know $A \neq \emptyset$, so $k \geq 1$. But D could be empty (that's what we want to show!) so there may be no V_i's.

If ~~any~~ there are any V_i's, let t be the smallest left endpoint of any of the V_i's.



Note that t is now in some U_i.

But this means $\exists\, x \in U_i \cap V_i$ which is impossible since x cannot be in both A & D.

Lemma (Homotopy lifting lemma)

Let $F: I \times I \longrightarrow S^1$ be a homotopy between the loops $F(t,0) = \alpha(t)$ & $F(t,1) = \beta(t)$ based at $(1,0)$. Suppose $\tilde{\alpha}(t)$ is a lift of α. Then we can lift F to the homotopy

$\tilde{F}: I \times I \longrightarrow S^1 \ni \tilde{F}(t,0) = \tilde{\alpha}(t)$, and

$p(\tilde{F}(t,s)) = F(t,s) \quad \forall\, t,s.$

(Note that $\tilde{F}$ ends at the path $\tilde{F}(t,1)$ which is a lift of $\beta(t)$.

Pf: The proof is similar to the path lifting lemma, But now we must cut $I \times I$ up into a finite # of squares so that each small square is taken into S^1 U or V

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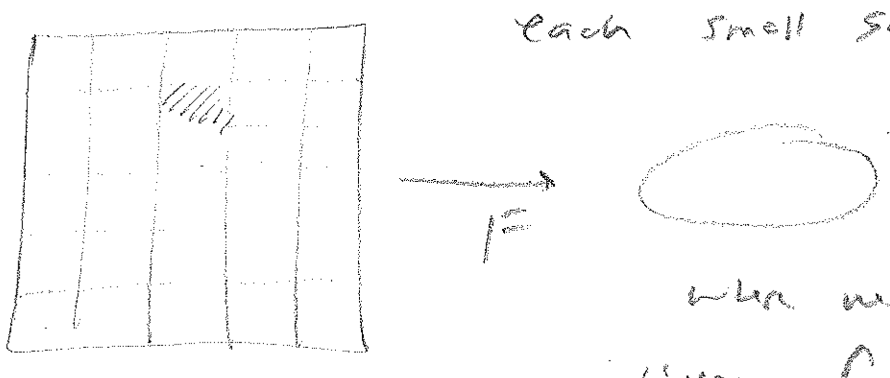

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when we can use the theorem for q to lift F. ∎

<u>Def</u> Let $\gamma_n : [0,1] \longrightarrow \mathbb{R}^1$ be the path from 0 to n given by $\gamma_n(t) = nt$.

<u>Thm</u> The map $\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1, (1,0))$ given by $\Phi(n) = [p \circ \gamma_n]$ is an isomorphism.

<u>Pf</u>: Note that γ_n is a path in $\mathbb{R}^1$ & $p \circ \gamma_n$ is a loop in S^1. $[p \circ \gamma_n]$ is the homotopy class of the loop $p \circ \gamma_n$ and is an element in $\pi_1(S^1, (1,0))$.

<u>Claim</u> Φ is a homomorphism. We need to show

$$\Phi(n+m) = \Phi(n)\,\Phi(m)$$

in otherwords

$$p \circ \gamma_{n+m} \sim (p \circ \gamma_n) \cdot (p \circ \gamma_m)$$

Define the product of the two paths γ_n & γ_m to be

$$(\gamma_n \cdot \gamma_m)(t) = \begin{cases} \gamma_n(2t) & 0 \le t \le \tfrac{1}{2} \\ n + \gamma_m(2t-1) & 0 \le t \le 1 \end{cases}$$

$$= \begin{cases} 2nt & 0 \le t \le \tfrac{1}{2} \\ n + m(2t-1) & \tfrac{1}{2} \le t \le 1 \end{cases}$$

Note that $(p \circ \gamma_n) \cdot (p \circ \gamma_m) = p \circ (\gamma_n \cdot \gamma_m)$

Next note that $\exists$ a straight-line homotopy

$\widetilde{F} : I \times I \longrightarrow \mathbb{R}^1$ between γ_{n+m} & $\gamma_n \cdot \gamma_m$

defined as

$$\widetilde{F}(t,s) = (1-s)\gamma_{n+m}(t) + s(\gamma_n \cdot \gamma_m)(t)$$

where $\widetilde{F}(t,0) = \gamma_{n+m}(t)$, $\widetilde{F}(t,1) = (\gamma_n \cdot \gamma_m)(t)$

& $\widetilde{F}(0,s) = 0 \quad \forall s$

& $\widetilde{F}(1,s) = n+m \quad \forall s$

Now let $F(t,s) = (p \circ \hat{F})(t,s)$.

Then F is a homotopy between

$p \circ \gamma_{n+m}$ & $p \circ (\gamma_n \cdot \gamma_m) = (p \circ \gamma_n) \cdot (p \circ \gamma_m)$.

Thus Φ is a homomorphism. $\boxtimes$

<u>Claim</u> Φ is onto.

Let $\alpha : [0,1] \to S^1$ be any loop based at $(1,0)$. $\exists$ a lift $\tilde{\alpha} : [0,1] \to \mathbb{R}^1$ $\ni$ $\tilde{\alpha}(0) = 0$.

Now $\tilde{\alpha}(1)$ is some integer, say n.

As before, $\exists$ homotopy $\hat{F} : I \times I \to \mathbb{R}^1$ between $\tilde{\alpha}$ and γ_n. Now $p \circ F$ is a homotopy between $\alpha = p \circ \tilde{\alpha}$ and $p \circ \gamma_n = \Phi(n)$. Thus Φ is onto.

<u>Claim</u> Φ is 1:1

We'll show the kernel of Φ is zero.

Suppose $\Phi(n)$ is the identity in $\pi_1(S^1, (1,0))$

ie $p \circ \gamma_n \sim e$, the trivial map $e : [0,1] \to S^1$ with $e(t) = (1,0)$ for all t.

Now $\tilde{e} : [0,1] \longrightarrow \mathbb{R} \ni \tilde{e}(t) = 0 \ \forall t$ is a lift of e.

By the homotopy lifting Lemma we can lift the homotopy to get a homotopy in $\mathbb{R}^1$ between $\tilde{e}$ and some path β that is a lift of $p \circ \gamma_n$. By the uniqueness of lifting we must have $\beta = \gamma_n$. Then $\tilde{e}$ is homotopic to γ_n, keeping 0 & 1 fixed. Hence $n = 0$.

Hence $\ker \Phi = \{0\}$ & Φ is 1:1.

$\blacksquare$