Math 177 HW 1 Due Tuesday, January 23

1. Show that every isometry $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a bijection. (One to one and onto.) Hence every isometry has an inverse.

2. An isometry is called *central* if it fixes 0. Show that every isometry is a central isometry followed by translation.

3. Thinking of $\mathbb{R}^n$ as a vector space over $\mathbb{R}$, show that every central isometry is a linear transformation, that is,

   (a) For all $x, y, \in \mathbb{R}^n$, we have $\phi(x + y) = \phi(x) + \phi(y)$.

   (b) For all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we have $\phi(\lambda x) = \lambda\phi(x)$.

   Hence every central isometry is given my multiplication by a matrix. Combined with the previous result, we now know that that every isometry is of the form $\phi(x) = Ax+b$ for some matrix A and some vector b.

4. Recall that a matrix A with real entries is called *orthogonal* if $A^T A = I$. Show that if A is orthogonal, then $\det A = \pm 1$. Show that $\phi(x) = Ax$ defines a central isometry if and only if A is an orthogonal matrix.

5. What are all 1×1 orthogonal matrices? Use this to show that every isometry of $\mathbb{R}^1$ is either a translation or a reflection.

6. Show that a 2×2 orthogonal matrix must be of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

   Show that the first matrix corresponds to rotation around the origin through the angle θ and the second to reflection through the line containing the origin that makes an angle of $\theta/2$ with the x-axis.

7. An isometry $\phi(x) = Ax + b$ is called *direct*, or *orientation preserving*, if $\det A = 1$ and *opposite*, or *orientation reversing*, if $\det A = -1$. Show that for $\mathbb{R}^2$, a direct isometry is either a rotation or a translation and that an opposite isometry is either a reflection or a glide reflection. (A *glide reflection* is reflection in a line ℓ followed by translation in a direction parallel to ℓ.)