

Math 177 HW 1 Due Tuesday, January 23

1. Show that every isometry $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a bijection. (One to one and onto.) Hence every isometry has an inverse.

Suppose $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isometry. If x and y are points with $\phi(x) = \phi(y)$, then $0 = d(\phi(x), \phi(y)) = d(x, y)$. Hence $x = y$ and ϕ is one to one.

To illustrate the main idea in showing that ϕ is onto, suppose that $n = 2$. Let y be any point in the range. We want to find a point x in the domain so that $\phi(x) = y$. Pick three non-colinear points in the domain, say $0 = (0, 0)$, $e_1 = (1, 0)$, and $e_2 = (0, 1)$. Let y_0, y_1 and y_2 be the images of these three points under ϕ , respectively. Now let $d_i = d(y, y_i)$ for $i = 1, 2, 3$. Because $n = 2$, it is not that hard to see that the point y is the only point located at these three distances from y_0, y_1 , and y_2 . This is a crucial part of the proof. Next, there must be a point x in the domain also located these three distances from $0, e_1$, and e_2 . Since ϕ preserves distances, it must take x to a point located these three distances from y_0, y_1 , and y_2 and because y is the only such point, we must have $\phi(x) = y$.

To make this proof really rigorous and easier to state in any dimension n , let's use Exercise 3. (Reading ahead, we can check that the proof of Exercise 3 does not depend on this exercise!). Let's also prove the following fact:

Claim 1: If $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isometry and T is any triangle in the domain, then T is congruent to its image $\phi(T)$.

Proof of Claim 1: The three vertices of T are taken to three points that maintain their same distance apart. Thus T is taken to a triangle with the same side-lengths. Hence T and its image are congruent.

Thinking of just two sides of a triangle, we see that if two vectors in the domain make an angle of α then their images are two vectors of the same length that also make an angle of α .

Suppose $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isometry and y is any point in the range. Let e_1, e_2, \dots, e_n be the standard basis in the domain. These vectors are all 1 unit long and all mutually perpendicular. Such a set of vectors is called orthonormal. Hence the origin 0 is taken to some point $\mathcal{O} = \phi(0)$ and the vectors are all taken to an orthonormal set of vectors based at \mathcal{O} . Let's *introduce* coordinates in the range by using \mathcal{O} as the origin and the set $\{\phi(e_1), \phi(e_2), \dots, \phi(e_n)\}$ as basis. Now y has coordinates with respect to this coordinate frame, say $y = (y_1, y_2, \dots, y_n)$. Let x be the point in the domain with the same set of coordinates. Now using Exercise 3, we have

$$\phi((y_1, y_2, \dots, y_n)) = \phi(y_1 e_1 + y_2 e_2 + \dots + y_n e_n) = y_1 \phi(e_1) + y_2 \phi(e_2) + \dots + y_n \phi(e_n) = y.$$

Thus ϕ is onto.

2. An isometry is called *central* if it fixes 0. Show that every isometry is a central isometry followed by translation.

Suppose $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isometry and $\mathcal{O} = \phi(0)$. Let T be the translation which takes \mathcal{O} to the origin. Now if we do ϕ and follow it by T , then the result, which we'll call C , is an isometry (because translation is an isometry and composition of isometries is an isometry). Moreover, C fixes the origin and so is central. Now the isometry T has an inverse, let's call it T^{-1} , which is translation in the opposite direction. Finally, we now have that ϕ is C followed by T^{-1} .

3. Thinking of \mathbb{R}^n as a vector space over \mathbb{R} , show that every central isometry is a linear transformation, that is,

- (a) For all $x, y \in \mathbb{R}^n$, we have $\phi(x + y) = \phi(x) + \phi(y)$.
- (b) For all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we have $\phi(\lambda x) = \lambda\phi(x)$.

Hence every central isometry is given by multiplication by a matrix. Combined with the previous result, we now know that every isometry is of the form $\phi(x) = Ax + b$ for some matrix A and some vector b .

Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be any central isometry. Let v and w be vectors. Consider the parallelogram with vertices $0, v, w$, and $v + w$. Because ϕ is an isometry, this will have to be taken to a congruent parallelogram with vertices $0 = \phi(0), \phi(v), \phi(w)$, and $\phi(v + w)$. But this means that $\phi(v + w) = \phi(v) + \phi(w)$.

Let v be any vector and λ any real number. We want to show that $\phi(\lambda v) = \lambda\phi(v)$. This is clearly true if $\lambda = 0$. Suppose then, that $0 < \lambda < 1$. Now the points $0, \lambda v$, and v are colinear with λv between 0 and v . So the distance from 0 to v is the sum of the distances from 0 to λv and from λv to v . These distances are preserved by ϕ . From this it follows that $\phi(\lambda v)$ must lie on the line segment joining 0 to $\phi(v)$. (If not, then we would have a triangle where the length of one side was equal to the sum of the lengths of the other two sides, which is not possible.) Moreover, it must be true that $\phi(\lambda v) = \lambda\phi(v)$. The other cases, for example where λ is 1, or greater than 1, etc., are handled similarly.

4. Recall that a matrix A with real entries is called *orthogonal* if $A^T A = I$. Show that if A is orthogonal, then $\det A = \pm 1$. Show that $\phi(x) = Ax$ defines a central isometry if and only if A is an orthogonal matrix.

Since $\det(AB) = \det(A)\det(B)$, $\det(A^T) = \det(A)$, and $\det(I) = 1$, it follows that if A is orthogonal, then $\det(A)^2 = 1$. Thus $\det(A) = \pm 1$.

Suppose $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a central isometry. Then we know it is a linear map, and hence given by multiplication by some matrix A . The i -th column of A is $\phi(e_i) = Ae_i$, where e_i is the standard basis element whose i -th coordinate is 1 while all other

coordinates are zero. Since an isometry preserves the lengths of vectors, we now see that every column of A is a unit vector. But an isometry also preserves the angle between vectors. Since e_i and e_j are perpendicular if $i \neq j$, we have that any pair of columns of A are perpendicular. But this means that A is orthogonal.

Conversely, suppose $\phi(v) = Av$ where A is an orthogonal matrix. We'll show first that for every vector v , both v and Av have the same length. Recall that $|v|^2 = v \cdot v = v^T v$. So,

$$|Av|^2 = (Av)^T(Av) = v^T A^T Av = v^T (A^T A)v = v^T I v = v^T v = |v|^2.$$

Hence $|Av| = |v|$. Now the distance between v and w is $|v - w|$. So, we want to show that $|Av - Aw| = |v - w|$. But $Av - Aw = A(v - w)$, so $|Av - Aw| = |A(v - w)| = |v - w|$.

5. What are all 1×1 orthogonal matrices? Use this to show that every isometry of \mathbb{R}^1 is either a translation or a reflection.

A 1×1 orthogonal matrix is either (1) or (-1) . Every isometry is a central one followed by translation. Hence every isometry is either $\phi(x) = 1 \cdot x + a$ or $\phi(x) = -1 \cdot x + a$. The first is translation by a . Now reflection through the point b is given by sending x to $2b - x$. (Check that the midpoint of x and $2b - x$ is b .) Hence $\phi(x) = -x + a = 2(a/2) - x$ is reflection through $a/2$.

6. Show that a 2×2 orthogonal matrix must be of the form

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}.$$

Show that the first matrix corresponds to rotation around the origin through the angle θ and the second to reflection through the line containing the origin that makes an angle of $\theta/2$ with the x -axis.

If A is orthogonal then each column is a unit vector and the two columns are perpendicular to each other. Any unit vector is equal to $(\cos \theta, \sin \theta)$ where θ is the angle the vector makes with the positive x axis. Thus the first column of A is $\begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$ and

the second column is $\begin{pmatrix} \cos(\theta \pm \pi/2) \\ \sin(\theta \pm \pi/2) \end{pmatrix}$ which is either $\begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$ or $\begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}$.

Thus A is either

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \text{ or } \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}.$$

It is clear that the first is rotation by θ . The second is rotation by θ followed by reflection across the line $y = (\tan \theta)x$. By examining the images of the standard basis vectors e_1 and e_2 , we see that the second is reflection through the line $y = (\tan(\theta/2))x$.

7. An isometry $\phi(x) = Ax + b$ is called *direct*, or *orientation preserving*, if $\det A = 1$ and *opposite*, or *orientation reversing*, if $\det A = -1$. Show that for \mathbb{R}^2 , a direct isometry is either a rotation or a translation and that an opposite isometry is either a reflection or a glide reflection. (A *glide reflection* is reflection in a line ℓ followed by translation in a direction parallel to ℓ .)

Using the previous exercise, we need to show that a rotation followed by a translation is a rotation and that a reflection followed by translation is a glide reflection. Consider first a rotation followed by translation: $\phi(x) = Ax + b$, where A is the matrix of the first kind given in the previous exercise. If this is to be a rotation about some point, then that point will be held fixed. So, is there an x so that $Ax + b = x$? This means that $(A - I)x = -b$ and we can solve this for x if and only if $A - I$ is invertible, or equivalently, $\det(A - I) \neq 0$. But

$$A - I = \begin{pmatrix} \cos \theta - 1 & -\sin \theta \\ \sin \theta & \cos \theta - 1 \end{pmatrix}$$

and has determinant $\cos^2 \theta - 2 \cos \theta + 1 + \sin^2 \theta = 2(1 - \cos \theta)$. This is zero only if $\theta = 0$, in which case we do only translation. Otherwise, if we do a nontrivial rotation followed by translation, then there is a unique fixed point. Thus we must have a rotation about the fixed point, as the only isometry of the plane with a single fixed point is rotation. Alternatively, we may have a reflection followed by translation. Let ℓ be the axis of the reflection and let A and B be two points on ℓ . The reflection does not move A and B after which they are moved to $A + b$ and $B + b$. (The translation is: x goes to $x + b$.) Let ℓ' be the line parallel to ℓ and lying midway between ℓ and the parallel line containing $A + b$ and $B + b$. Then it is not hard to see that the final isometry is a glide reflection with axis ℓ' .