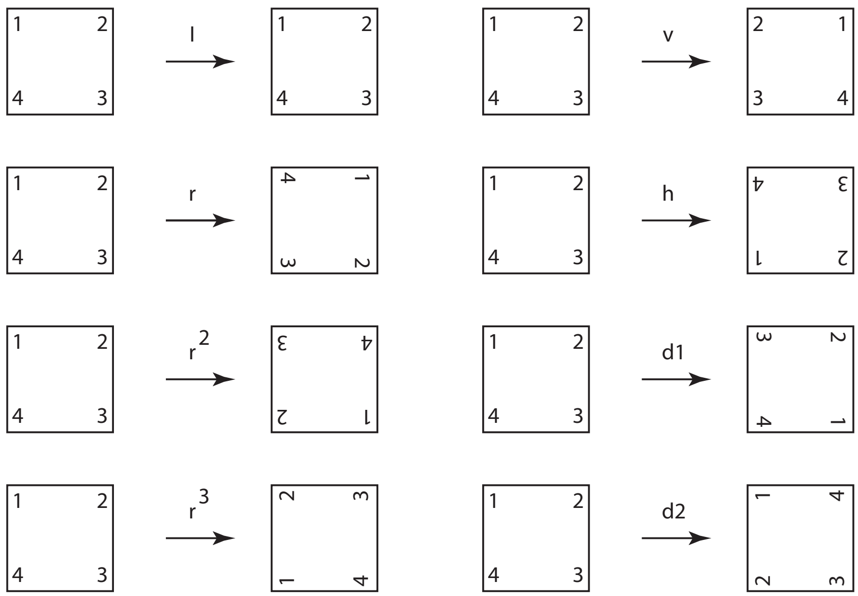
Math 177 HW 2 Solutions

8. The eight symmetries of the square are shown below. The rotation is denoted r and the "flips" are called $v, h, d1$ and $d2$ for "vertical", "horizontal", "diagonal (one)" and "diagonal (two)", respectively.



(a) Complete the following multiplication table of the symmetry group of the square. This groups is called the *dihedral group of order 8* and is denoted D_4. More generally, the symmetry group of a regular n-gon is called the dihedral group D_n, and has $2n$ elements.

Solution:

	I	r	r^2	r^3	v	$h = vr^2$	$d1 = vr$	$d2 = vr^3$
I	I	r	r^2	r^3	v	$h = vr^2$	$d1 = vr$	$d2 = vr^3$
r	r	r^2	r^3	I	$d2 = vr^3$	$d1 = vr$	v	$h = vr^2$
r^2	r^2	r^3	I	r	$h = vr^2$	v	$d2 = vr^3$	$d1 = vr$
r^3	r^3	I	r	r^2	$d1 = vr$	$d2 = vr^3$	$h = vr^2$	v
v	v	$d1 = vr$	$h = vr^2$	$d2 = vr^3$	I	r^2	r	r^3
$h = vr^2$	$h = vr^2$	$d2 = vr^3$	v	$d1 = vr$	r^2	I	r^3	r
$d1 = vr$	$d1 = vr$	$h = vr^2$	$d2 = vr^3$	v	r^3	r	I	r^2
$d2 = vr^3$	$d2 = vr^3$	v	$d1 = vr$	$h = vr^2$	r	r^3	r^2	I

(b) Is the symmetry group of the square commutative?

Solution: No. $vr \neq rv$.

9. Continuing with the symmetry group of the square, show that r and v will generate the group. That is, show that every element of the group can be expressed in terms of r and v alone. In fact, show that every element of the group if equal to one of the following: $I, r, r^2, r^3, v, vr, vr^2$ and vr^3. (In general, for any n, the dihedral group D_n can be generated by the rotation and a single flip across one axis of bilateral symmetry.)

   Solution: Notice that each element of the group corresponds to one of the corners being labeled 1 and then having the sequence $1, 2, 3, 4$ arranged either clockwise or counter-clockwise. Thus I, r, r^2, and r^3 produce all possible elements with clockwise labeling. If we do v, we change to the counter-clockwise labeling. Thus v, vr, vr^2, and vr^3 produce all possible elements with counter-clockwise labeling. Finally, it is easy to check that $d1 = vr, h = vr^2$, and $d2 = vr^3$.

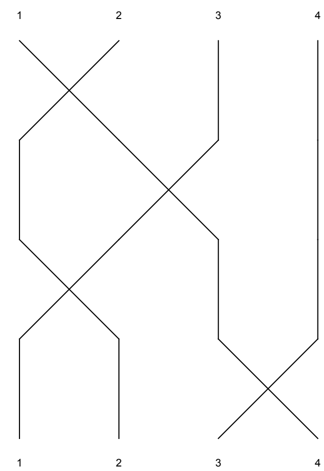
10. Using r and v as generators, draw the Cayley graph of D_4. Use two different colors for the edges, one color for r and one color for v. HINT: It will be helpful to use the multiplication table that you made for D_4, and even more helpful if you replace $h, d1$ and $d2$ in that table with their representations as vr, or vr^2, or vr^3.

    Solution: The graph is shown in Figures 6.16. 6.17, and 6.18 in our text book, *Groups and Their Graphs.*

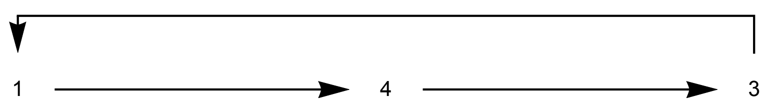
Permutations

A *permutation* of n things is a bijection $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. There are $n!$ possible permutations. There are various ways to record a permutation. Since a permutation is a function, we can just list its values on each of the inputs. So, for example with $n = 3$, we can list $\{\sigma(1), \sigma(2), \sigma(3)\}$. The six possible permutations are now $\{1, 2, 3\}$, $\{1, 3, 2\}$, $\{2, 1, 3\}$, $\{2, 3, 1\}$, $\{3, 1, 2\}$, and $\{3, 2, 1\}$. The first of these is the identity function since $\sigma(i) = i$ for $i = 1, 2, 3$. Since permutations are functions, we can compose them, and it is easy to check that doing so turns the set of all permutations into a group, called the *symmetric* group of n things and denoted by S_n.

Permutations can also be depicted graphically by drawing *braids.* The following braid diagram depicts the permutation $\sigma = \{4, 2, 1, 3\}$. The numbers 1, 2, 3, 4 are written across the top and the bottom of the braid and the function is thought of as taking the 4 numbers on top to the 4 numbers on the bottom by following the braid "strings" from top to bottom. The string that starts at 1 on the top ends at 4 on the bottom, so $\sigma(1) = 4$. To multiply two permutations, $\sigma\tau$ we simply stack the braid for σ on top of the braid for τ.



A better way to record a permutation is to write it as a product of *cycles*. In the example in the braid figure we see that 1 is taken to 4, 4 is taken to 3, and 3 is taken back to 1. Separately, 2 is taken to 2, which means that 2 is held fixed. We can depict this as the *3-cycle* shown below:



We don't use 2 in the drawing because 2 is not moved. We denote the cycle as $(1\,4\,3)$ and read it as "1 goes to 4 which goes to 3 which goes back to 1." We could also denote it as $(4\,3\,1)$ (which is "4 goes to 3 goes to 1 goes to 4") or $(3\,1\,4)$ (which is "3 goes to 1 goes to 4 goes to 1"). Note that if we reverse the arrows, then we would write $(1\,3\,4)$ or $(3\,4\,1)$ or $(4\,1\,3)$, all of which denote the same permutation. Moreover, the permutation $(1\,3\,4)$ is the inverse to $(1\,4\,3)$ since every number is sent back to where it came from. The permutation $(1\,3\,4)$ is called a *3-cycle* because the cycle has three numbers in it.

The identity permutation does not move any numbers so does not have any cycles. Rather than write it as nothing, or as the capital letter I, we usually write it as the 1-cycle (1).

A permutation may have more than one cycle. For example, $\sigma \in S_5$ might send 1 to 2, 2 to 4, 3 to 5, 4 to 1, and 5 to 3. This consists of two *disjoint* cycles: the 3-cycle $(1\,2\,4)$ and the 2-cycle $(3\,5)$. The cycles are disjoint because they have no numbers in common. It is easy to see that σ is the product $(1\,2\,4)(3\,5)$ or $(3\,5)(1\,2\,4)$. Thinking about braids, we see that *disjoint cycles commute*.

11. Write all six elements of S_3, the set of permutations of 1, 2, and 3, as both braid figures, and in cycle form. Do the same for the 24 elements of S_4.

    Solution: See Figures 1 to 5.

12. Fill in the following multiplication table of S_3. Here the elements are represented in cycle form. Warning: there is no particular reason to believe that multiplication of permutations is commutative, that is, independent of the order of multiplication. So don't just assume that (12) times (123) is the same as (123) times (12) and skip filling in BOTH entries in the multiplication table.

Solution:

	(1)	(12)	(13)	(23)	(123)	(132)
(1)	(1)	(12)	(13)	(23)	(123)	(132)
(12)	(12)	(1)	(123)	(132)	(13)	(23)
(13)	(13)	(132)	(1)	(123)	(23)	(12)
(23)	(23)	(123)	(132)	(1)	(12)	(13)
(123)	(123)	(23)	(12)	(13)	(132)	(1)
(132)	(132)	(13)	(23)	(12)	(1)	(123)

Does multiplication of permutations in S_3 turn out to be commutative?

Solution: No. $(12)(13) \neq (13)(12)$.

13. A *transposition* is a permutation consisting of a single 2-cycle. In other words, a transposition is a permutations that trades two things and keeps all other things fixed. In S_3 there are three transpositions: $(1\,2), (1\,3)$, and $(2\,3)$. What are all the transpositions in S_4?

Solution: The transposition in S_4 are $(1\,2), (1\,3), (1\,4), (2\,3), (2\,4)$, and $(3\,4)$.

14. Every permutation has an inverse because it is a bijection. If σ is represented by a braid, then turning the braid upside down will give the braid of its inverse.

    (a) List all 6 elements of S_3 together with their inverses.
    Solution: See Figure 1.

    (b) List all 24 elements of S_4 together with their inverses.
    Solution: See Figures 2 to 5.

    (c) What is the inverse of a transposition?
    Solution: Itself.

    (d) If a permutation is equal to its own inverse, does it have to be a transposition? (Look in S_4.)
    Solution: No. The permutation $(1\,2)(3\,4)$ is not a transposition, but is equal to its own inverse.

    (e) What is the inverse of the 3-cycle $(a\,b\,c)$? What is the inverse of the 4-cycle $(a\,b\,c\,d)$? In general, what is the inverse of the n-cycle $(a_1\,a_2\,\ldots\,a_n)$?
    Solution: Then inverse of the n-cycle $(a_1\,a_2\,\ldots\,a_n)$ is the n-cycle $(a_n\,a_{n-1}\,\ldots\,a_1)$ or, if we want to write a_1 first, $(a_1\,a_n\,a_{n-1},\ldots\,a_2)$

15. Prove that every permutation is a product of transpositions. Thus, the transpositions generate S_n. In fact, we only need transpositions of the form $(k\,k+1)$. So, S_3 is generated by $(1\,2)$ and $(2\,3)$. We don't need $(1\,3)$. (Hint: Use the braid diagram of a permutation to see this.) Using the braid diagrams for the elements of both S_3 and S_4, write all the element of S_3 and S_4 as products of transpositions.

    Solution: Let σ be any permutation and consider a braid diagram of σ. It is possible to draw horizontal lines through the braid diagram so that between each consecutive pair of horizontal lines, there is only one crossings. This choice of lines now factors σ into a product of transposition, each between a pair of adjacent strings. Thus transpositions of the form $(i\,j)$, where $|i-j|=1$, generate S_n.

    Each element of S_3 and S_4 is written as a product of transpositions of the form $(i\,i{+}1)$ in Figures 1 to 5.

16. Draw the Cayley graph of S_3 using the generators $(1\,2)$ and $(2\,3)$. Draw the Cayley graph of S_4 using the generators $(1\,2)$, $(2\,3)$, and $(3\,4)$.

    Solution: These graphs are shown in Figures 6 and 8.

17. Show that S_3 is generated by $(1\,2)$ and $(1\,2\,3)$. Draw the Cayley graph using these generators. Show that $(1\,2\,3)$ and $(1\,2\,3\,4)$ generate S_4. Using these as generators, draw the Cayley graph of S_4.

    Solution: Writing out the multiplication tables of both D_3 and S_3, we see that the two groups are isomorphic. In fact, under the isomorphism, r and f correspond to $(1\,2\,3)$ and $(1\,2)$, respectively. Thus the Cayley graph of S_3 using the generators $(1\,2\,3)$ and $(1\,2)$ is the same as the Cayley graph of D_3 with generators r and f. This graph is on page 48 of the book *Groups and Their Graphs*. The graph of S^4 is shown in Figure 7.

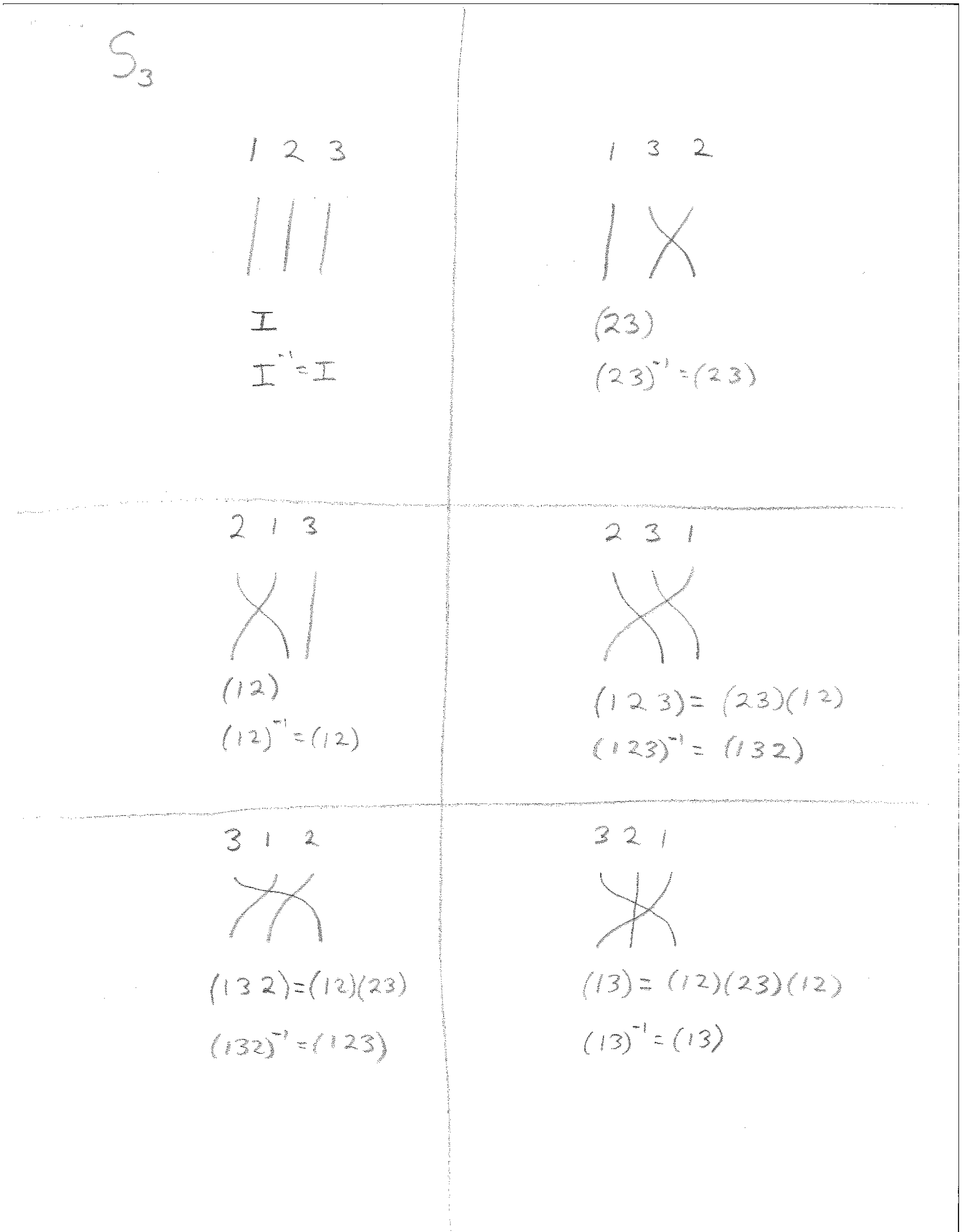


Figure 1: The six elements of S_3.

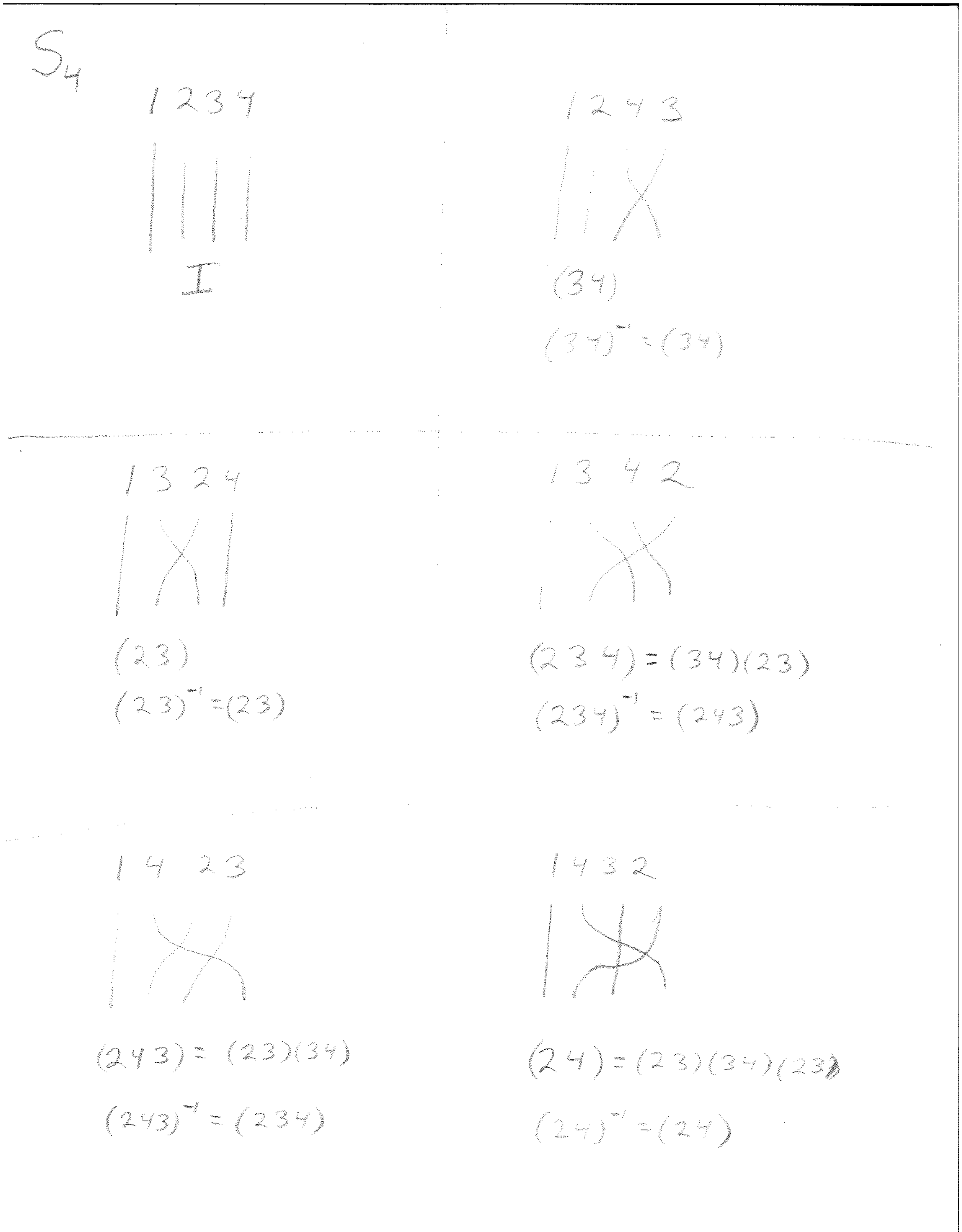


Figure 2: Six elements of S_4.

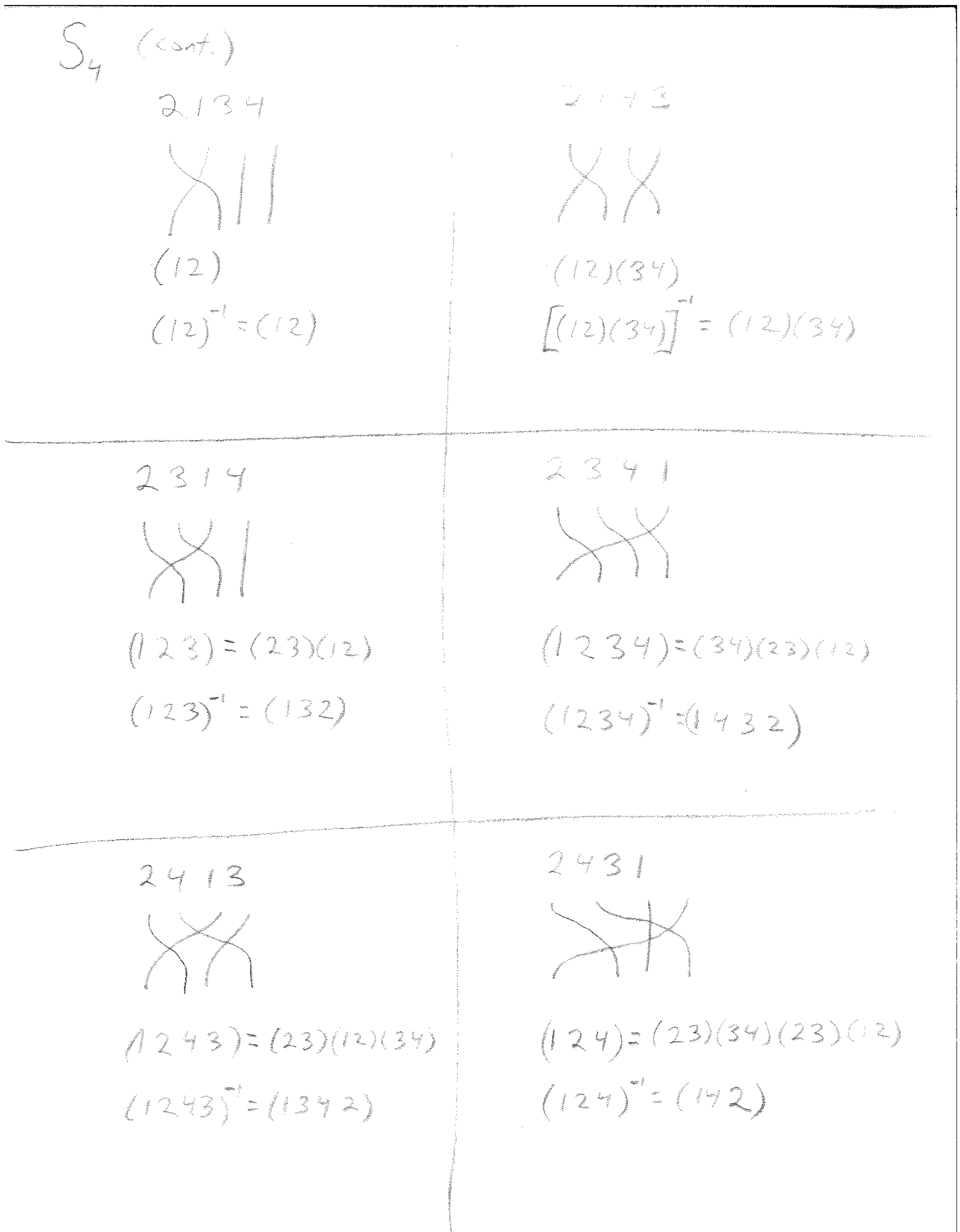


Figure 3: Six more elements of S_4.

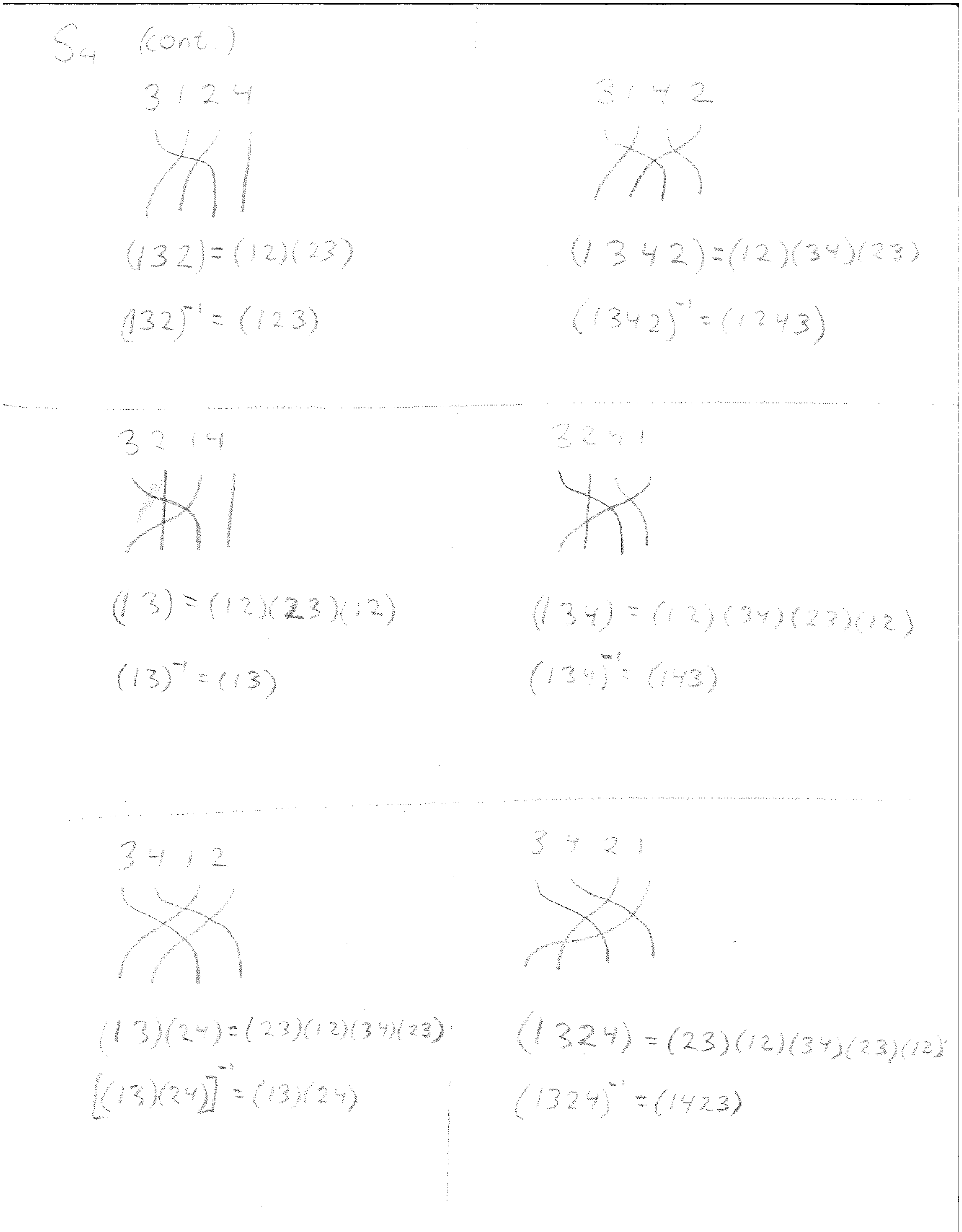


Figure 4: Six more elements of S_4.

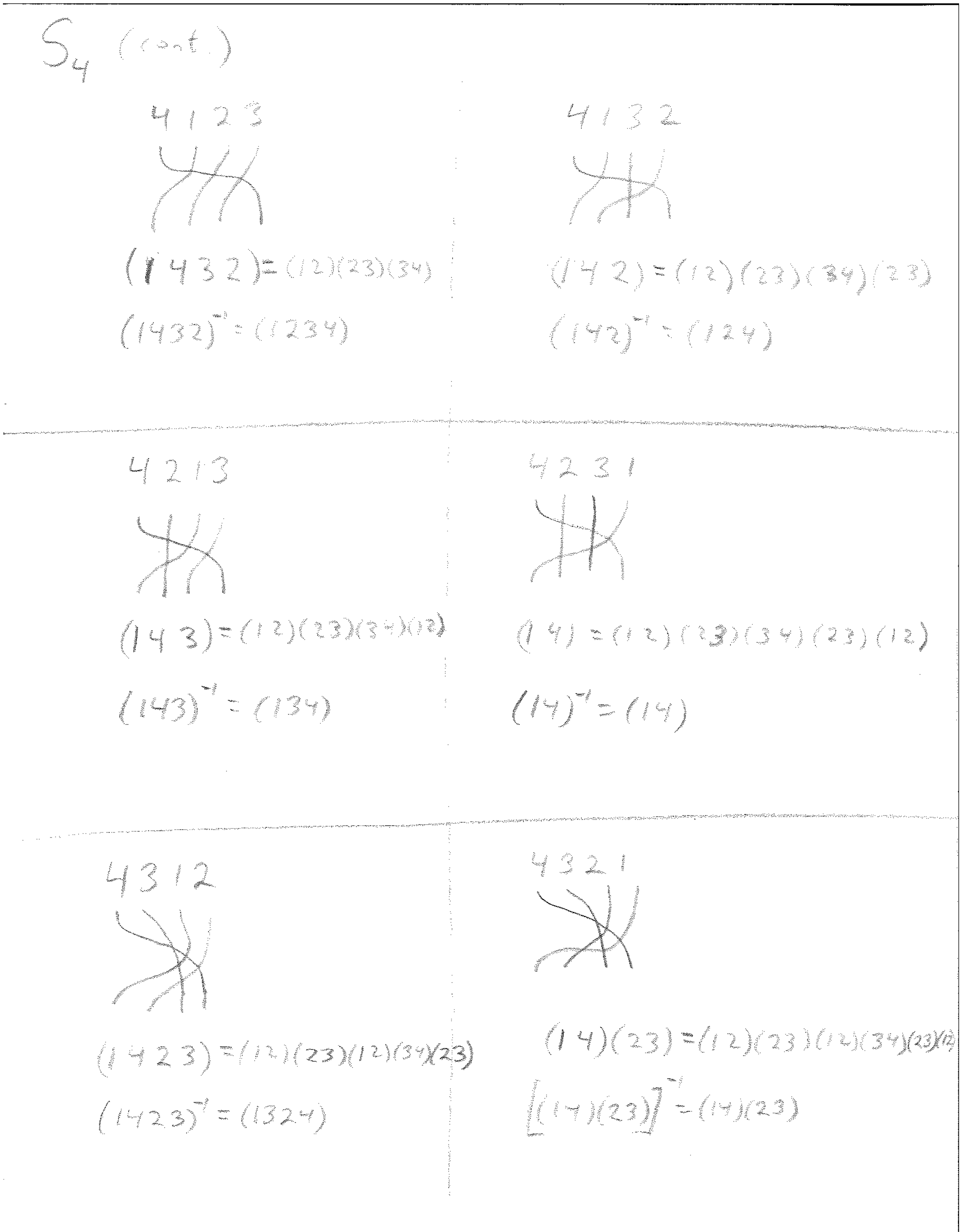


Figure 5: Six more elements of S_4.

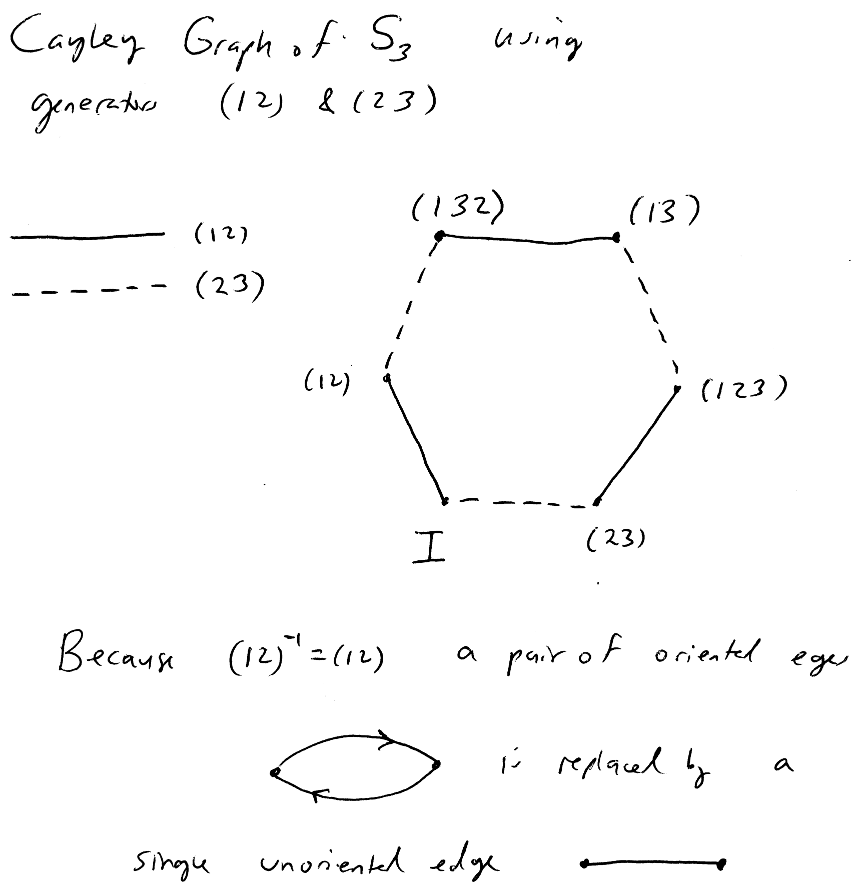


Figure 6: The Cayley graph of S_3 using the generators (1 2) and (2 3).

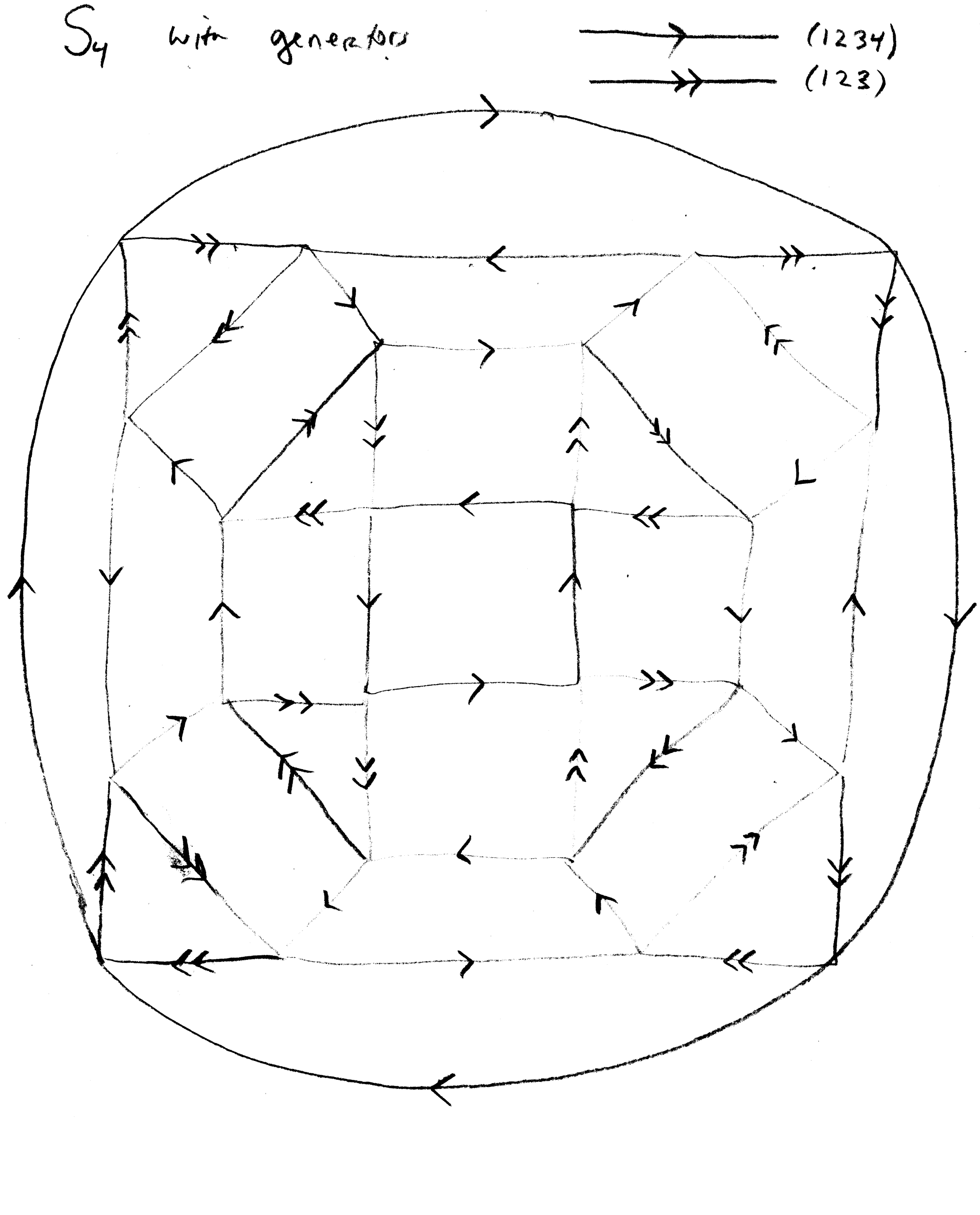


Figure 7: The Cayley graph of S_4 using the generators $(1\,2\,3)$ and $(1\,2\,3\,4)$.

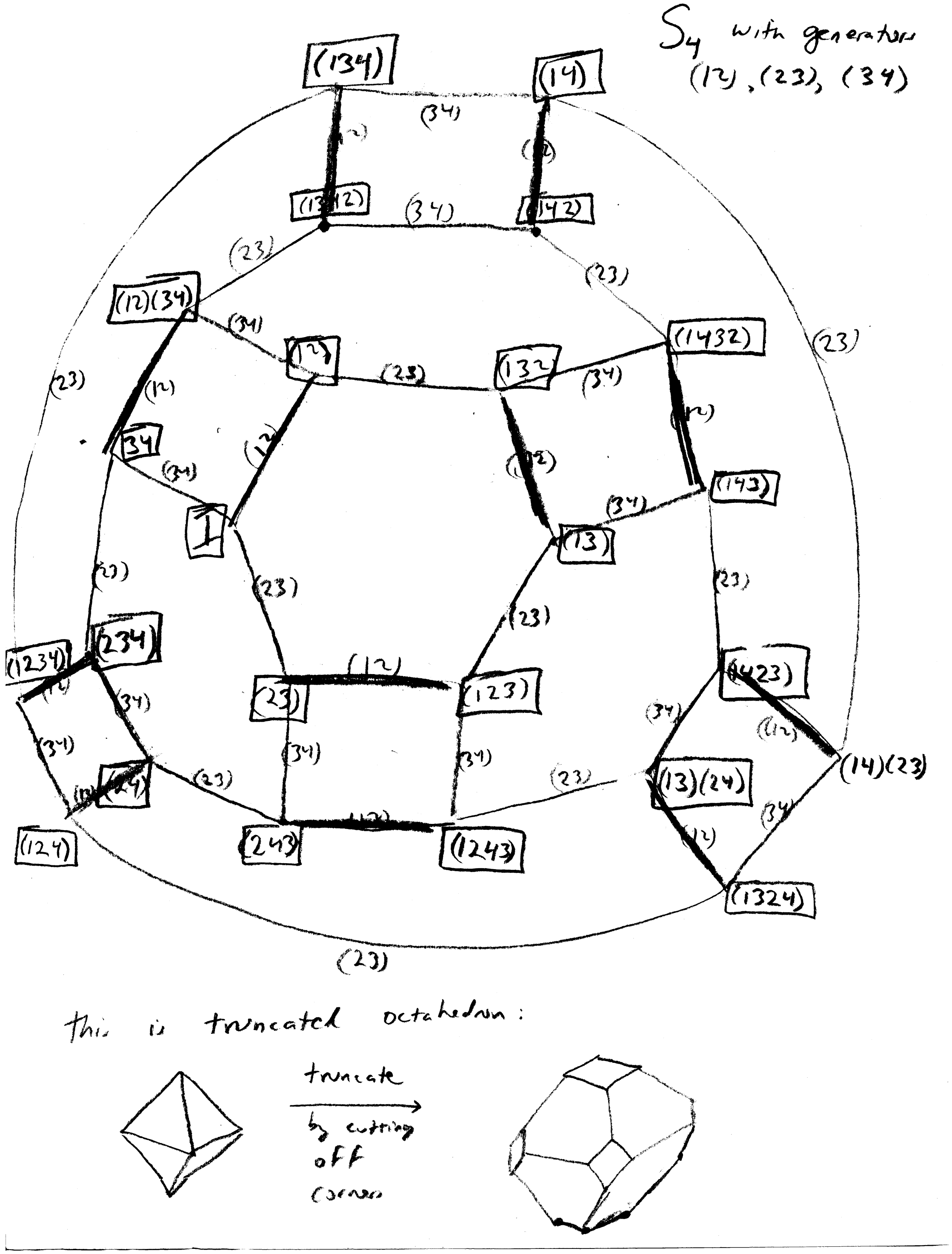


Figure 8: The Cayley graph of S_4 using the generators (1 2), (2 3), and (3 4).