

Math 177 HW 3 Due Friday, Feb 9 at noon

18. Let G be a group and S a subset of G . Let H be the set of all elements of G that can be expressed as products of elements in S and their inverses.
- (a) Show that H is a subgroup of G and that H is generated by S .
 - (b) Show that H is the smallest subgroup of G that contains S . That is, if H' is any subgroup of G that contains S , then H is contained in H' .
19. Suppose that G is a group and H_1 and H_2 are subgroups of G . Show that $H_1 \cap H_2$ is a subgroup of G . Give an example of a group G that contains two subgroups whose union is not a subgroup.
20. Prove or disprove this statement: Every subgroup of a cyclic group is cyclic.
21. Let \mathbb{Z} be the group of integers under addition. What are all subgroups of \mathbb{Z} ?
22. Let $n > 1$ be given. Find all subgroups of C_n , the cyclic group of order n . (Warning: The answer depends on whether n is prime or not.)
23. Given any group G , the *center* of G , denoted $Z(G)$, is defined as all elements of G that commute with every element of G . That is,

$$Z(G) = \{g \in G \mid gh = hg \text{ for all } h \in G\}.$$

- (a) Show that $Z(G)$ is a subgroup of G .
 - (b) Show that $Z(G) = G$ if and only if G is Abelian.
 - (c) Show that $Z(G)$ is always a normal subgroup of G .
24. Let $n > 1$ be given. Let r and f be the isometries of the regular n -gon that are counter-clockwise rotation through an angle of $2\pi/n$, and reflection through a line containing the center of the n -gon and a vertex of the n -gon, respectively. We may now write all the elements of D_n in terms in r and f as follows:

$$D_n = \{1, r, r^2, \dots, r^{n-1}, f, fr, \dots, fr^{n-1}\}.$$

- (a) Show that $r^k f = fr^{n-k}$ for all $0 < k < n$.
- (b) Assume that n is an odd prime. Find all subgroups of D_n . (Hint: Consider all subgroups generated by the subset S , where S has one element, or two elements, or three elements, etc. For example, what if S has two elements, one of which is a rotation, r^k , and one of which is a flip, fr^j ?)
- (c) Continuing to assume that n is an odd prime, what is the center $Z(D_n)$?
- (d) Find all subgroups of D_4 . Four is not prime, so things are a bit different.

- (e) What is the center of D_4 ?
25. Show that any subgroup H of a group G that has index two is normal. (The *index* of H in G is the number of right cosets of H in G , which is always the same as the number of left cosets of H in G .) This is Exercise 52 in our book and the solution is in the back of the book! Try to do it first yourself. Although in our book, the result is only stated in the case where G is finite.