Math 177 HW 3 Solutions

18. Let G be a group and S a nonempty subset of G. Let H be the set of all elements of G than can be expressed as products of elements in S and their inverses. More specifically,
$$H = \{g_1 g_2 \dots g_n \,|\, n \in \mathbb{N}, g_i \in S \text{ or } g_i^{-1} \in S\}.$$

    (a) Show that H is a subgroup of G and that H is generated by S.

    Solution: Because S is nonempty, so it H. Next, suppose that $h_1 = g_1 g_1 \dots g_n$ is in H. Then $h_1^{-1} = g_n^{-1} \dots g_2^{-1} g_1^{-1}$ is in H since each g_i or g_i^{-1} is in S. Finally, if $h_1 = g_1 g_1 \dots g_n$ and $h_2 = f_1 f_1 \dots f_m$ is in H, then $h_1 h_2 = h_1 = g_1 g_1 \dots g_n f_1 f_1 \dots f_m$ is in H. Therefore, H is a subgroup.

    (b) Show that H is the smallest subgroup of G that contains S. That is, if H' is any subgroup of G that contains S, then H is contained in H'.

    Solution: Suppose that H' is any subgroup that contains S. Because H' is a subgroup, it is closed under multiplication and taking inverses. Hence, any word made out of elements of S or elements whose inverses are in S must be in H'. Therefore, $H \subset H'$.

19. Suppose that G is a group and H_1 and H_2 are subgroups of G. Show that $H_1 \cap H_2$ is a subgroup of G. Give an example of a group G that contains two subgroups whose union is not a subgroup.

    Solution: The identity is in both H_1 and H_2 and hence in $H_1 \cap H_2$. Thus $H_1 \cap H_2$ is nonempty. Next, if $h \in H_1 \cap H_2$, then $h \in H_1$ and $h \in H_2$. Because these are subgroups, $h^{-1} \in H_1$ and in $h^{-1} \in H_2$ and so, $h^{-1} \in H_1 \cap H_2$. Finally, if $h_1, h_2 \in H_1 \cap H_2$, then $h_1, h_2 \in H_1$ and $h_1, h_2 \in H_2$. Because these are subgroups, we now have $h_1 h_2 \in H_1$ and $h_1 h_2 \in H_2$. Hence $h_1 h_2 \in H_1 \cap H_2$. Thus $H_1 \cap H_2$ is a subgroup.

20. Prove or disprove this statement: Every subgroup of a cyclic group is cyclic.

    Solution: Let $G = \mathbb{Z}$, $H_1 = 2\mathbb{Z}$, and $H_2 = 3\mathbb{Z}$. Now 2 and 3 are in $H_1 \cup H_2$ but 2+3=5 is not in the union. Hence the union is not a subgroup.

21. Let $\mathbb{Z}$ be the group of integers under addition. What are all subgroups of $\mathbb{Z}$?

    Solution: Since $\mathbb{Z}$ is cyclic, every subgroup of $\mathbb{Z}$ is cyclic. Thus we just need to see which subgroup each integer generates. Zero generates the trivial subgroup. If n is a nonzero integer, then n generates the subgroup $n\mathbb{Z}$.

22. Let $n > 1$ be given. Find all subgroups of C_n, the cyclic group of order n. (Warning: The answer depends on whether n is prime or not.)

    Solution: Since C_n is cyclic, every subgroup of C_n is cyclic. Thus we need only see what each element of C_n generates. Suppose that C_n is generated by g and let g^k

be any nontrivial element of C_n. Let $d = \gcd(k, n)$. If $d = 1$, then g^k generates all of C_n. Otherwise, g^k generates a subgroup of order d. Thus, there are subgroups of order d for every divisor d of n.

23. Given any group G, the *center* of G, denoted $Z(G)$, is defined as all elements of G that commute with every element of G. That is,

$$Z(G) = \{g \in G \,|\, gh = hg \text{ for all } h \in G\}.$$

  (a) Show that $Z(G)$ is a subgroup of G.

  Solution: The identity is in $Z(G)$ since the identity commutes with every element of G. Thus $Z(G)$ is nonempty. Next, if $z \in Z(G)$ and $g \in G$, then , then $zg^{-1} = g^{-1}z$ Taking the inverse of each side of this equation gives $gz^{-1} = z^{-1}g$. Hence $z^{-1} \in G$. Finally, suppose that z_1 and z_2 are in the center. Let g be any element of G. Now $z_1z_2g = z_1gz_2 = gz_1z_2$ showing that z_1z_2 is in the center.

  (b) Show that $Z(G) = G$ if and only if G is Abelian.

  Solution: The group G is abelian if an only if every element commutes with every element, which is exaclty what it means for the center to be the entire group.

  (c) Show that $Z(G)$ is always a normal subgroup of G.

  Solution: Let g be any element of G. Now

$$\begin{aligned} gZ(G) &= \{gz \,|\, z \in Z(G)\} && (1) \\ &= \{zg \,|\, z \in Z(G)\} && (2) \\ &= Z(G)g && (3) \end{aligned}$$

  Thus $Z(G)$ is normal, since any left coset of $Z(G)$ is a right coset of $Z(G)$.

24. Let $n > 1$ be given. Let r and f be the isometries of the regular n-gon that are counter-clockwise rotation through an angle of $2\pi/n$, and reflection though a line containing the center of the n-gon and a vertex of the n-gon, respectively. We may now write all the elements of D_n in terms in r and f as follows:

$$D_n = \{1, r, r^2, \ldots, r^{n-1}, f, fr, \ldots, fr^{n-1}\}.$$

  (a) Show that $r^kf = fr^{n-k}$ for all $0 < k < n$.

  Solution: Number the vertices of the n-gon $v_0, v_1, \ldots, v_{n-1}$ in counter-clockwise order. Assume that the reflection f is through an axis containing the vertex v_0. Now the rotation r takes v_i to v_{i+1} (where we think of the vertices as labeled mod n) and f takes v_i to v_{n-i}. It is now easy to verify that both rf and fr^{-1} take v_i to v_{n-i+1} and so are the same symmetry.

(b) Assume that n is an odd prime. Find all subgroups of D_n. (Hint: Consider all subgroups generated by the subset S, where S has one element, or two elements, or three elements, etc. For example, what if S has two elements, one of which is a rotation, r^k, and one of which is a flip, fr^j?)

Solution: First consider nontrivial cyclic subgroups H. If H is generated by fr^k, then $H = \{1, fr^k\}$ because $(fr^k)^2 = fr^k fr^k = ffr^{-k}r^k = 1$. If H is generated by r^k, then $H = \{1, r, r^2, \ldots, r^{n-1}\}$ since n is prime and so $\gcd(k, n) = 1$. If H is generated by two nontrivial elements x and y, suppose $x = r^k$ and $y = fr^j$. Now some power of x is r^{-j} and $r^{-j}y = r^{-j}fr^j = f$. Now that H contains both r and f, we see that $H = D_n$. If H is generated by two rotations, then by virtue of containing a single rotation, $H = \{1, r, r^2, \ldots, r^{n-1}\}$. If H is generated by two reflections, then their product gives a rotation. Thus H now contains all rotations, plus at least two reflections and so $H = D_n$. If H is generated by three or more elements, then again we get that $H = \{1, r, r^2, \ldots, r^{n-1}\}$ or $H = D_n$.

(c) Continuing to assume that n is an odd prime, what is the center $Z(D_n)$?

Solution: Suppose that r^k is in the center. Then we need $r^k fr^j = fr^j r^k$ for all j. But this means that $fr^{j-k} = fr^{j+k}$ which means $r^{j-k} = r^{j+k}$ which means $r^{2k} = 1$. Thus we need $2k = 0$ or $2k = n$. Since n is an odd prime, we must have $r = 0$. If fr^k is in the center, then in order to commute with r^j, we now need that $r^{2j} = 1$ and $j = 0$. Hence the only rotation that fr^k commutes with is the identity. Hence the center is trivial.

(d) Find all subgroups of D_4. Four is not prime, so things are a bit different.

Solution: The subgroups are $\{1\}$, $\{1, r^2\}$, $\{1, f\}$, $\{1, fr\}$, $\{1, fr^2\}$, $\{1, fr^3\}$, $\{1, r, r^2, r^3\}$, and D_4.

(e) What is the center of D_4?

Solution: The center is $\{1, r^2\}$.

25. Show that any subgroup H of a group G that has index two is normal. (The *index* of H in G is the number of right cosets of H in G, which is always the same as the number of left cosets of H in G.) This is Exercise 52 in our book and the solution is in the back of the book! Try to do it first yourself. Although in our book, the result is only stated in the case where G is finite.

Solution: Because H has index 2, any nontrivial coset of H is exactly the complement of H in G. Thus the left cosets of H are H and the complement of H and these two sets are exactly the right cosets of H as well. So, the left collection of left cosets is exactly the same as the collection of right cosets and so H is normal.