Math 177 HW 4 Due Friday, Feb 16 at noon

26. Suppose G is a group where every element has order 2. Show that G is Abelian.

27. Find all groups of order 6.

28. Suppose p is a prime. Let G be the set of all 2 by 2 matrices with entries which are integers mod p that have determinant not equal to zero mod p.

    (a) Using matrix multiplication as the operation (and, again, doing all integer arithmetic mod p), show that G is a group. This group is called $GL(2, \mathbb{Z}/p\mathbb{Z})$. The "GL" stands for "general linear" group, the "2" for the size of the matrices, and finally $\mathbb{Z}/p\mathbb{Z}$ are the integers mod p.

    (b) Let $SL(2, \mathbb{Z}/p\mathbb{Z})$ be the subset of $GL(2, \mathbb{Z}/p\mathbb{Z})$ of matrices with determinant equal to 1 mod p. Show that $SL(2, \mathbb{Z}/p\mathbb{Z})$ is a subgroup of $GL(2, \mathbb{Z}/p\mathbb{Z})$. The group $SL(2, \mathbb{Z}/p\mathbb{Z})$ is called the "special linear" group.

    (c) Show that $SL(2, \mathbb{Z}/2\mathbb{Z}) = GL(2, \mathbb{Z}/2\mathbb{Z})$ and has order 6. Where does this group appear in your answer to Exercise 27?

    (d) What are the orders of $SL(2, \mathbb{Z}/p\mathbb{Z})$ and $GL(2, \mathbb{Z}/p\mathbb{Z})$ for any prime p? This might be hard. Here are the orders for small p, from which you might be able to guess at the answer and then use that to help prove the general result. These results were obtained by writing a computer program to generate all possible matrices for each p and then count the ones that had non-zero determinant or determinant 1.

p	$\lvert SL(2, \mathbb{Z}/p\mathbb{Z})\rvert$	$\lvert GL(2, \mathbb{Z}/p\mathbb{Z})\rvert$
2	6	6
3	24	48
5	120	480
7	336	2016
11	1320	13200
13	2184	26208
17	4896	78336
19	6840	123120

29. Suppose that H is a subgroup of the group G.

    (a) Declare that $x, y \in G$ are *equivalent*, written $x \sim y$, if $x^{-1}y \in H$. Show that this is an equivalence relation on G.

    (b) Show that the equivalence classes are exactly the right cosets of H

    (c) What equivalence relation will have as its equivalence classes the left cosets of H?

30. Show that half the permutations in S_n are even and half are odd. Thus $|A_n| = n!/2$.

31. Show that S_n can be generated by $(1\,2)$ and the n-cycle $(1\,2\,3\ldots\,n)$.

32. Show that every subgroup of an Abelian group is normal.

33. If n is a natural number, let $n\mathbb{Z}$ denote all integer multiples of n.

    (a) Show that $n\mathbb{Z}$ is a normal subgroup of $\mathbb{Z}$.

    (b) What are the right cosets of $n\mathbb{Z}$ and what is the index of $n\mathbb{Z}$ in $\mathbb{Z}$?

    (c) What is $n\mathbb{Z} \cap m\mathbb{Z}$?

    (d) Is every subgroup of $\mathbb{Z}$ of the form $n\mathbb{Z}$ for some n?