Math 177 HW 5 Due Friday, Feb 23 at noon

34. Recall that if G and H are groups, their *direct product* is defined as the set

$$G \times H = \{(g, h) \mid g \in G, h \in H\},$$

with the operation given by

$$(a, b)(c, d) = (ac, bd).$$

When is the product of cyclic groups cyclic?

35. Find a presentation of A_4 by taking the following steps:

    (a) Let $a = (1\,2\,3)$ and $b = (1\,2\,4)$. Show that a and b generate A_4.

    (b) Draw the Cayley graph Γ of A_4 using the generators a and b. This graph is planar, so make sure you draw it in the plane without any edges crossing each other.

    (c) Choose a maximal tree T in Γ. The graph Γ will have 24 edges and the tree T will have have 11 edges. This leaves 13 edges in Γ that are not in T.

    (d) Label each face of Γ with the word that describes the loop given by going around the boundary of the face. (A face is a region of the plane in the complement of Γ.) Call the set of words you get R.

    (e) For each of the 13 edges in Γ that are not in T, write down the relation determined by that edge. Show that each relation is a consequence of the relations in R.

    (f) What is the final presentation, with generators a and b, of A_4?

36. The goal of this exercise is to find a presentation for S_4, by building on the work that we did in Exercise 35.

    (a) Let $c = (1\,2)$. Show that $S_4 = A_4 \cup (1\,2)A_4$.

    (b) Imagine taking two copies of the the graph Γ and placing one in the plane $z = 0$ in $\mathbb{R}^3$ and the other directly above it in the plane $z = 1$. Keep the one in the plane $z = 0$ labeled exactly as the Cayley graph of A_4. Suppose we connect each vertex in the upper graph to the vertex immediately below it in the lower graph with two edges: one labeled $(1\,2)$ going up and a second one labeled $(1\,2)$ going down. Can you label the upper graph so that the whole thing is now the Cayley graph of S_4? Are the elements of the coset $(1\,2)A_4$ the vertices of the upper graph?

(c) Using the Cayley graph of S_4 that you just constructed, choose a maximal tree $\mathcal{T}$ that consists of the tree T that you chose for A_4 in Exercise 35, a copy of that tree directly above it, and one vertical edge labeled c connecting the lower tree with the upper tree. (Take c to connect (1) and (1 2).)

(d) Write down the relations that come from the faces of the lower graph, the upper graph, and every "vertical face". Call this set R.

(e) Show that every relation that comes from an edge in $S_4 - \mathcal{T}$ is a consequence of the relations in R.

(f) What is the presentation you obtain for S_4?

37. Find a presentation of D_n with two generators: one a rotation and one a reflection.

38. Find a presentation of D_n with two reflections as generators.

39. Let G be the group defined by the presentation

$$G = \langle x, y \,|\, xy = yx \rangle.$$

Show that G is isomorphic the direct product $\mathbb{Z} \times \mathbb{Z}$.

40. Show that the groups given by the presentations

$$\langle a, b \,|\, a^2 = b^3 \rangle$$

and

$$\langle x, y \,|\, xyx = yxy \rangle$$

are isomorphic.