Math 177 HW 5 Due Friday, Feb 23 at noon

34. Recall that if G and H are groups, their *direct product* is defined as the set

$$G \times H = \{(g,h) \,|\, g \in G, h \in H\},$$

with the operation given by

$$(a,b)(c,d) = (ac, bd).$$

When is the product of cyclic groups cyclic?

Solution: Claim 1: The product of two finite cyclic groups is cyclic if and only if their orders are relatively prime.
Proof: We need to know some fact about relatively prime integers. This fact will suffice: If n and m are positive integers, then

$$\operatorname{lcm}(n,m)\gcd(n,m) = nm$$

where $\gcd(n,m)$ is the greatest common divisor of n and m and $\operatorname{lcm}(n,m)$ is the least common multiple of n and m. Moreover, two positive integers are called *relatively prime* if their greatest common divisor is 1, or alternatively, their least common multiple is their product.

Suppose that G is cyclic of order n, generated by x and H is cyclic of order m, generated by y. Consider the element $(x,y) \in G\times H$ and suppose that $(x,y)^d = (1,1)$. Hence d divides the order of the group, which is nm. But $(x,y)^d = (x^d, y^d)$. Hence $(x,y)^d = (1,1)$ if and only if $x^d = 1$ and $y^d = 1$. But $x^d = 1$ if and only if d is a multiple of n and $y^d = 1$ if and only if d is a multiple of m. Thus $(x,y)^d = (1,1)$ if and only if d is a common multiple of n and m. If n and m are relatively prime, then their least common multiple is nm and hence $nm \le d$. Therefore we must have $d = nm$ and we see that $G \times H$ is cyclic with generator (x,y). If instead, n and m are not relatively prime, then $\operatorname{lcm}(n,m) < nm$. It now follows that no element of $G \times H$ can generate the group, for if (x^k, y^j) is any element, then $(x^k, y^j)^{\operatorname{lcm}(n,m)} = (1,1)$. Hence every element has order less than nm. $\square$

Claim 2: If at least one of G or H is an infinite cyclic group, then $G \times H$ is cyclic if and only if the other factor is the trivial group.
Proof: Suppose that G is infinite cyclic, generated by x, and H is cyclic, generated by y. If $G \times H$ is cyclic, then there is an element (x^j, y^k) that generates the group. But powers of this element will produce power of x^j in the first position. In order for the product to be cyclic we would need $j = \pm 1$. So, let's assume that $j = \pm 1$. Now $(x^{\pm 1}, y^k)^d = (x^{\pm d}, y^{kd})$. Notice that the only element we can get of the form $(1,z)$ is when $d = 0$ and hence $z = 1$. Thus if H is nontrivial, we cannot generate all of the

product. If H is trivial, then $(x, 1)$ will in fact generate the product and hence the product is cyclic. A similar argument can be given in the case where H is infinite cyclic. $\square$

Combining the two claims, the final answer is: The product of cyclic groups G and H is cyclic if and only if they are both finite with orders that are relatively prime, or one is infinite and the other is trivial.

35. Find a presentation of A_4 by taking the following steps:

    (a) Let $a = (1\,2\,3)$ and $b = (1\,2\,4)$. Show that a and b generate A_4.
    Solution: First note that a and b generate all the 3-cycles:

    $$\begin{array}{ll} (1\,3\,4) = (1\,2\,3)(1\,4\,2)(1\,3\,2) & (1\,4\,3) = (1\,2\,3)(1\,2\,4)(1\,3\,2) \\ (2\,3\,4) = (1\,2\,3)(1\,4\,2) & (2\,4\,3) = (1\,2\,4)(1\,3\,2) \end{array}$$

    The group A_4 is the set of all even permutations in S_4. Hence every element of A_4 is a product of an even number of transpositions. Grouping such a product in pairs, we see that it suffices to show that the product of any two transpositions can be expressed in terms of 3-cycles (which in turn can each be expressed in terms of a and b). Consider the product $(i\,j)(k\,l)$. If i, j, k, and l are all distinct, then

    $$(i\,j)(k\,l) = (i\,k\,l)(i\,k\,j).$$

    If this is not the case, then it must be that one of i or j is equal to one of k or l, and the four letters represent three distinct numbers. (We must have $i \neq j$ and $k \neq l$ and if the four letters represent only two distinct numbers, then we have $(i\,j)(i\,j) = 1$.) Without loss of generality, we may assume that $j = k$. But now

    $$(i\,j)(j\,l) = (i\,l\,j).$$

    Thus any permutation that is the product of an even number of transpositions can be written as a product of 3-cycles, which in turn can be written entirely in terms of a and b. Hence, a and b generate A_4. $\square$

    (b) Draw the Cayley graph Γ of A_4 using the generators a and b. This graph is planar, so make sure you draw it in the plane without any edges crossing each other.

    (c) Choose a maximal tree T in Γ. The graph Γ will have 24 edges and the tree T will have have 11 edges. This leaves 13 edges in Γ that are not in T.

    (d) Label each face of Γ with the word that describes the loop given by going around the boundary of the face. (A face is a region of the plane in the complement of Γ.) Call the set of words you get R.

(e) For each of the 13 edges in Γ that are not in T, write down the relation determined by that edge. Show that each relation is a consequence of the relations in R.

(f) What is the final presentation, with generators a and b, of A_4?
Solution: The Cayley graph is shown in Figure 1 with a maximal tree T shaded heavily. The faces give the relators: $a^3, b^3, abab$. Each edge not in T gives a relation. For example, consider the the edge labeled a leaving the vertex labeled 1. This gives the relation $a(baabaaba) = 1$. But this relator is a consequence of the face relators as follows:

$$\begin{aligned}
abaabaaba &= aba^{-1}ba^{-1}ba && \text{replace } aa \text{ with } a^{-1} \\
&= abbabbbabba && \text{replace } a^{-1} \text{ with } bab \\
&= ab^{-1}abbbab^{-1}a && \text{replace } bb \text{ with } b^{-1} \\
&= ab^{-1}aab^{-1}a && \text{delete } bbb \\
&= ab^{-12}a^{-1}b^{-1}a && \text{replace } aa \text{ with } a^{-1} \\
&= aaa && \text{replace } b^{-1}a^{-1}b^{-1} \text{ with } a \\
&= 1 && \text{delete } aaa
\end{aligned}$$

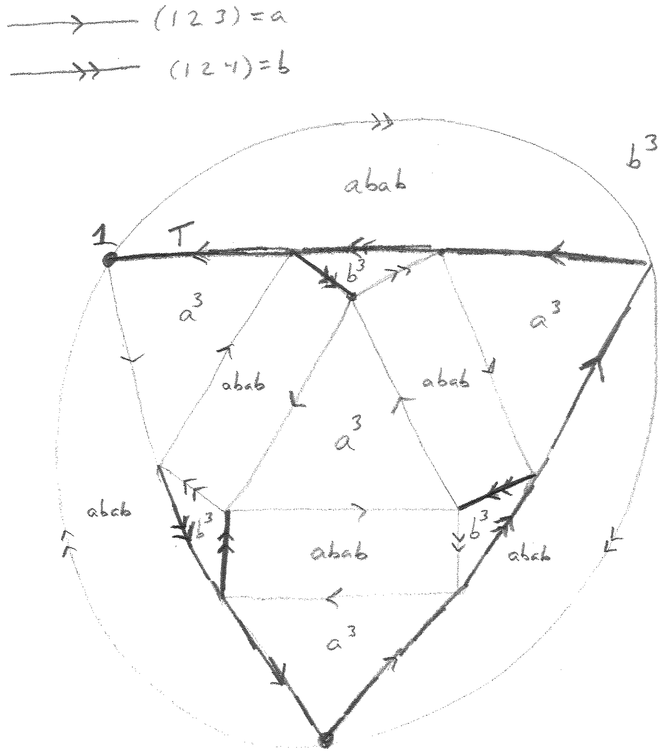
Similarly, the relations obtained from other edges of the Cayley graph that are not in T can be shown to be consequences of the three face relations.
So, the presentation of A_4 is

$$A_4 = \langle a, b \,|\, a^3 = 1, b^3 = 1, (ab)^2 = 1 \rangle.$$

36. The goal of this exercise is to find a presentation for S_4, by building on the work that we did in Exercise 35.

(a) Let $c = (1\,2)$. Show that $S_4 = A_4 \cup (1\,2)A_4$.
Solution: The subgroup A_4 has only one coset, the odd permutations, which is exactly what $(1\,2)A_4$ is.

(b) Imagine taking two copies of the the graph Γ and placing one in the plane $z = 0$ in $\mathbb{R}^3$ and the other directly above it in the plane $z = 1$. Keep the one in the plane $z = 0$ labeled exactly as the Cayley graph of A_4. Suppose we connect each vertex in the upper graph to the vertex immediately below it in the lower graph with two edges: one labeled $(1\,2)$ going up and a second one labeled $(1\,2)$ going down. Can you label the upper graph so that the whole thing is now the Cayley graph of S_4? Are the elements of the coset $(1\,2)A_4$ the vertices of the upper graph?



The Cayley graph of A_4 with generators a & b and a maximal tree T heavily shaded.

Figure 1: The Cayley graph of A_4 with generators $a = (1\,2\,3)$ and $b = (1\,2\,4)$.

(c) Using the Cayley graph of S_4 that you just constructed, choose a maximal tree $\mathcal{T}$ that consists of the tree T that you chose for A_4 in Exercise 35, a copy of that tree directly above it, and one vertical edge labeled c connecting the lower tree with the upper tree. (Take c to connect (1) and $(1\,2)$.)

(d) Write down the relations that come from the faces of the lower graph, the upper graph, and every "vertical face". Call this set R.

(e) Show that every relation that comes from an edge in $S_4 - \mathcal{T}$ is a consequence of the relations in R.

(f) What is the presentation you obtain for S_4?
Solution: Let Γ_0 be a copy of Γ in the plane $z = 0$ and place a copy of Γ directly above it in the plane $z = 1$ and call it Γ_1. Change Γ_1 by reversing the orientation of every edge. If v is a vertex of Γ_0 let $\bar{v}$ be the vertex of Γ_1 directly above v. Connect Γ_0 to Γ_1 by adding an edge labeled c going from each vertex v in Γ_0 to $\bar{v}$ and also add an edge labeled c going from $\bar{v}$ to v. Call this big graph Σ. A rectangular loop has now been constructed in Σ for every edge e in Γ_0 by doing $ecec^{-1}$. This gives us relations $acac^{-1}$ and $bcbc^{-1}$. We also have the relation $c^2 = 1$ which allows us to rewrite $acac^{-1}$ as $acac$ and $bcbc^{-1}$ as $bcbc$. Claim: Any loop in Σ is a consequence of the relators

$$R = \{a^3, b^3, c^2, abab, acac, bcbc\}.$$

Once we prove the claim, we get the presentation for S_4 of

$$S_4 = \langle a, b, c \,|\, a^3, b^3, c^2, abab, acac, bcbc\rangle.$$

To prove the claim, fix a vertex in Γ_0 can call it 1. Let $\bar{1}$ be the vertex directly above 1. We want to show that any loop in Σ based at 1 is a consequence of R. We will assume this the case for any loop in Γ_0 based at 1, and therefore for any loop in Γ_1 based at $\bar{1}$. Suppose that α is any loop based at 1. Using the relators $acac$ and $bcbc$ we can change α to a loop that only goes up and down on the vertical edges between 1 and $\bar{1}$. Now we see that α is of the following form: a loop based at 1 in Γ_0, then c, then a loop in based at $\bar{1}$ in Γ_1, then c, then a loop in Γ_0, then c, and so on. But each loop, in either Γ_0 or Γ_1, can be changed to the empty word using $\{a^3, b^3, abab\}$. Thus α can be changed to the empty word using R.

37. Find a presentation of D_n with two generators: one a rotation and one a reflection.
Solution: Let r and f be the rotation and reflection that generate D_n. Draw the Cayley graph with these generators. We get an n-gon oriented clockwise, a larger concentric n-gon oriented counter-clockwise with edges that are all labeled r and then

edges labeled f that connect the inner vertices to the outer vertices. The graph is planar and the faces give the relators: r^n, f^2, and $rfrf$. Thus a presentation is

$$\langle r, f \mid r^n = 1, f^2 = 1, rfrf = 1\rangle.$$

$\square$

38. Find a presentation of D_n with two reflections as generators.
Solution: Let f_1 and f_2 be two reflections whose axis make an angle of $\frac{2\pi}{2n}$. Then their product is the rotation r through the angle $\frac{2Pi}{n}$. Thus f_1 and f_2 generate D_n. Draw the Cayley graph with these generators. We get a graph whose $2n$ vertices can be arranged as the regular vertic es of a regular $2n$-gon. Between a pair of adjacent vertices is a 2-gon with edges f_1 and f_1. Then betwen the next adjacent pair of vertices is a 2-gon with edges f_2 and f_2, etc. The graph is planar and the faces give the relators: f_1^2, f_2^2, and $(f_1f_2)^n$. Thus a presentation is

$$\langle f_1, f_2 \mid f_1^2 = 1, f_2^2 = 1, (f_1f_2)^n = 1\rangle.$$

$\square$

39. POSTPONED. Let G be the group defined by the presentation

$$G = \langle x, y \mid xy = yx\rangle.$$

Show that G is isomorphic the direct product $\mathbb{Z} \times \mathbb{Z}$.

40. POSTPONED. Show that the groups given by the presentations

$$\langle a, b \mid a^2 = b^3\rangle$$

and

$$\langle x, y \mid xyx = yxy\rangle$$

are isomorphic.