Math 177 HW 6 Due Friday, Mar 23 at noon

Suppose that G is a group generated by the subset X and R is any subset of G. Then a word w in X is a *consequence* of R if w can be changed into the empty word in X (the identity in G) by a finite sequence of the following two operations:

1. Insert or delete a subword of the form xx^{-1} or $x^{-1}x$ where $x \in X$.

2. Insert or delete the subword r or r^{-1}, where $r \in R$.

If w_1 and w_2 are words in X, then we say that w_1 and w_2 are R-*equivalent*, denoted $w_1 \sim_R w_2$, if w_1 can be changed into w_2 by a finite sequence of the above two operations. Thus, a word is a consequence of R if and only if it is R-equivalent to the identity. It is easy to show that R-equivalence is in fact an equivalence relation. (Every word in X is equivalent to itself by doing a sequence of no operations; if we can go from w_1 to w_2 via a sequence of the allowable steps, then we can reverse the sequence to go from w_2 to w_1; if we can go from w_1 to w_2 by a sequence of allowable steps and then from w_2 to w_3 by a sequence of allowable steps, then by stringing all the steps together, we can go from w_1 to w_3.)

39. Let G be a group and let R be a subset of G. Show that each of the following two descriptions defines a normal subgroup of G and that they are the same.

    (a) The intersection of all normal subgroups of G that contain R.

    (b) The set of elements

    $$\{g_1 r_1^{\epsilon_1} g_1^{-1} g_2 r_2^{\epsilon_2} g_2^{-1} \cdots g_n r_n^{\epsilon_n} g_n^{-1} \mid n \in \mathbb{N}, g_i \in G, r_i \in R, \epsilon_i \in \{-1, 1\}\}.$$

    Moreover, the normal subgroup of G defined by any one of the above descriptions is the smallest normal subgroup containing R, meaning that it is contained in any normal subgroup containing R.

40. Let G be a group generated by the subset X and let R be a subset of G. Let N be the smallest normal subgroup containing R. The goal of this exercise is to show that N is exactly the set of consequences of R.

    (a) Let $\overline{R}$ be the set of consequences of R. Show that $\overline{R}$ is a normal subgroup containing R. Thus $N \subset \overline{R}$ since N is the smallest normal subgroup containing R.

    (b) Suppose that w_1 and w_2 are words in X and that $r \in R$. Show that if $w_1 w_2 \in N$, then $w_1 r^{\pm 1} w_2 \in N$. This may be the hardest part. You need to use all of the following facts:

        i. $wN = Nw$ for any w.

ii. $wN = N$ if and only if $w \in N$.

iii. $w_1 N = w_2 N$ if and only if $w_1^{-1} w_2 \in N$.

(c) Show that if $w_1 \in N$ and $w_1 \sim_R w_2$, then $w_2 \in N$.

(d) Show that if $w \sim_R 1$, then $w \in N$.

(e) Show that $\overline{R} \subset N$. Hence $N = \overline{R}$.

41. Let G be a group generated by the subset X and let R be a subset of G. Let $\overline{R}$ be the smallest normal subgroup containing R and let w_1 and w_2 be words in X. Prove that the following are equivalent:

(a) The words w_1 and w_2 are in the same coset of $\overline{R}$.

(b) The words w_1 and w_2 are R-equivalent.

(c) The word $w_1^{-1} w_2$ is R-equivalent to 1.

42. Let G be the group defined by the presentation

$$G = \langle x, y \,|\, xyx^{-1}y^{-1} = 1 \rangle.$$

This means that $G = F(\{x, y\})/\overline{R}$ where $F(\{x, y\})$ if the free group on $\{x, y\}$ and $\overline{R}$ is the smallest normal subgroup of $F(\{x, y\})$ containing the set $R = \{xyx^{-1}y^{-1}\}$. The goal of this exercise is to show that the function $f : \mathbb{Z} \times \mathbb{Z} \to G$ given by $F((n, m)) = x^n y^m \overline{R}$ is an isomorphism.

(a) Show that if w is any word in x and y, then $w \sim_R x^n y^m$, where n and m are the total exponent sums of x and y, respectively, in w.

(b) Show that the words w_1 and w_2 are R-equivalent if and only if the total exponent sum of both x and y is the same for both w_1 and w_2.

(c) Show that f is a homomorphism.

(d) Show that f is a bijection and hence an isomorphism.

43. If V is a finite dimensional vector space and W is a subspace, then the dimension of W is less than or equal to the dimension of V. The analog of this for free groups is NOT true: If F is a free group of rank n then F can have a subgroup that is a free group of larger rank! The object of this exercise is to give such an example. This is exercise 15 in Chapter 1 of “Presentations of Groups” by Johnson.

Let H be the subset of reduced words of even length in $F = F(\{x, y\})$.

(a) Show that H is is a subgroup generated by

$$a = x^2, b = xy, c = xy^{-1}.$$

(b) If w is any word in $\{a, b, c\}$, then we can convert it to a word in x and y by replacing a with x^2, b with xy, and c with xy^{-1}. There are thirty 2-letter reduced words in $\{a, b, c\}$. Show that after converting each to a word in $\{x, y\}$, and reducing if necessary, the resulting reduced words each have length at least 2.

(c) More generally, show that any reduced word of length ℓ in $\{a, b, c\}$, when converted to a word in $\{x, y\}$, will result in a reduced word of at least length ℓ.

(d) Use Proposition 3 in Chapter 1 of Johnson to conclude that H is a free group of rank 3.