Math 177 HW 8 Due ??

50. Suppose that X and Y are spaces (that is, each is a subset of some Euclidean space) and that f and g are continuous functions from X to Y. We say that f is *homotopic* to g if there exists a continuous function $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ for all $x \in X$ and $H(x, 1) = g(x)$ for all $x \in X$. We write $f \sim g$ and call H a *homotopy* between f and g. We can jazz this up a little bit as follows: Suppose A is a subset of X and furthermore that $H(a, s) = f(a) = g(a)$ for all $a \in A$ and for all $s \in I$. In this case we say that f is homotopic to g *relative to* A and we write $f \sim g$ rel A. This definition extends our definition of homotopic loops. In the case of loops, we have $X = I$, $A = \{0, 1\}$, and we require that f and g are loops, that is $f(0) = f(1) = g(0) = g(1)$.

    Show that $\sim$ is an equivalence relation on the set of continuous functions from X to Y.

51. Suppose that f_1 and f_2 are homotopic maps from X to Y and that g_1 and g_2 are homotopic maps from Y to Z. Show that the composition $g_1 \circ f_1$ is homotopic to the composition $g_2 \circ f_2$.

52. Suppose that $f : X \to Y$ is a continuous map from the space X to the space Y taking the point $x_0 \in X$ to the point $y_0 = f(x_0) \in Y$. We can use f to obtain a map $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by defining f_* as $f_*([\alpha]) = [f \circ \alpha]$. Show that:

    (a) f_* is well-defined.

    (b) f_* is a homomorphism.

    (c) If $f, g : X \to Y$ are homotopic maps relative to x_0, then $f_* = g_*$.

    (d) If $f : X \to Y$ and $g : Y \to Z$ and $z_0 = g(y_0)$ and $y_0 = f(x_0)$, then $(g \circ f)_* = g_* \circ f_*$ (where these are maps from $\pi_1(X, x_0)$ to $\pi_1(Z, z_0)$).

53. Suppose that X and Y are spaces. A function $f : X \to Y$ is a *homeomorphism* if f is a continuous bijection and f^{-1} is also continuous. A cool way to say this is that f is a "bicontinuous bijection." Notice that $f \circ f^{-1} : Y \to Y$ is equal to the identity function 1_Y on Y and $f^{-1} \circ f : X \to X$ is equal to the identity function 1_X on X (by definition of the inverse of a function). We can relax this and consider functions $f : X \to Y$ and $g : Y \to X$ such that the compositions $f \circ g$ and $g \circ f$ are not necessarily the identity, but just homotopic to the identity. That is, we want $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$. In this case we say that both f and g are *homotopy equivalences* and we say that X and Y are *homotopy equivalent.*

    (a) Show that homotopy equivalence is an equivalence relation on the set of spaces.

(b) Show that if $f : X \to Y$ is a homotopy equivalence then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.

54. Suppose that X is a space and that A is a subset of X. A *retraction* from X to A is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in X$. We say that A is a *retract* of X.

    (a) Show that the following are retractions:

        i. Let X be any space and $x_0 \in X$ any point in X. Let $r : X \to x_0$ be the map that sends every point of X to x_0.

        ii. Let X and Y be any spaces and let $r : X \times Y \to X$ be defined as $r((x, y)) = x$.

    (b) If $r : X \to A$ is a retraction sending x_0 to a_0, show that $r_* : \pi_1(X, x_0) \to \pi_1(A, a_0)$ is a surjection.

    (c) Let $B^2 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$ and let S^1 be the subset of B^2 defined as $S^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$. Show that there cannot be a retraction from B^2 to S^1.

55. Let X be a space, A a subset, and $i : A \to X$ the inclusion map $i(a) = a$. We say that A is a *deformation retract* of X and that X *deformation retracts* to A if there is a retraction $r : X \to A$ and a homotopy between the identity function 1_X on X and $i \circ r$.

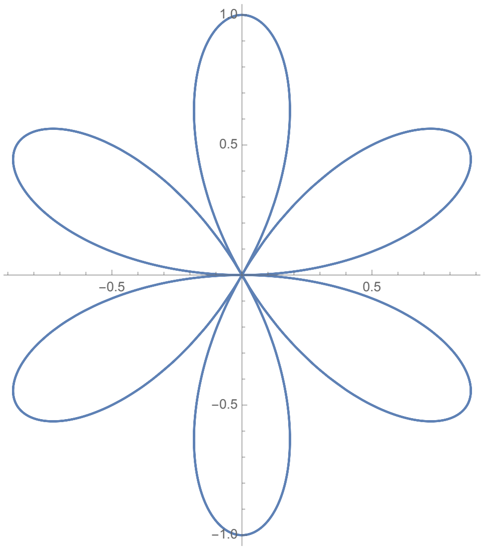
    (a) Suppose that X is a convex subset of $\mathbb{R}^n$ and that x_0 is any point in X. Show that X deformation retracts to x_0. A space that deformation retracts to a point is called *contractible.*

    (b) Suppose that X and Y are spaces, Y is contractible, and y_0 is a point in Y. Show that $X \times Y$ deformation retracts to $X \times \{y_0\}$.

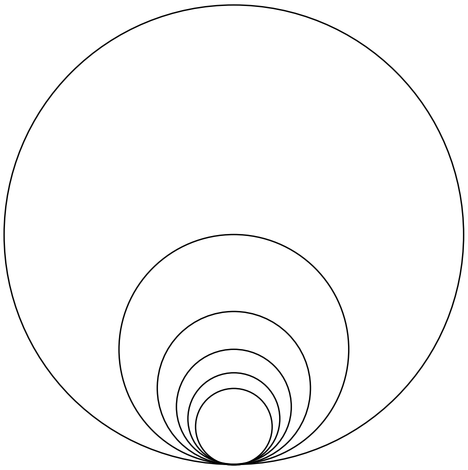
    (c) Suppose that T is a finite connected tree contained in $\mathbb{R}^2$. That is, T is a connected graph with a finite number of vertices and edges that contains no loops. Show that T is contractible.

56. Suppose that X deformation retracts to A, with retraction $r : X \to A$. Show that r is a homotopy equivalence.

57. A *bouquet of n circles* is a subset of $\mathbb{R}^2$ consisting of n "circles" such that each circle contains the origin and the intersection of all the circles is the origin. A bouquet of six circles is shown below:



Alternatively, if you really want to use geometrically round circles, we can picture the bouquet as seen below:



Another way to think about a bouquet of n circles is that it is a graph with one vertex and n edges, each of which is a loop with both of its endpoints at the single vertex.

Let Γ be any connected graph with v vertices and e edges. Show that Γ is homotopy equivalent to a bouquet of $e - v + 1$ circles.