Math 177 HW 9 Due May 7

Oriented knots can be depicted with *oriented knot diagrams* as shown in Figure 1. Here five knots are depicted. Their names are 0_1(unknot), 3_1(trefoil), 4_1(figure eight), 5_1(pentafoil), and 5_2(twist knot). Associated to every knot is the fundamental group of its complement. If two knots are really the same knot, then their associated groups will be isomorphic. The goal of this exercise set is to study the group of a knot and to use these groups to show that these five knots are all different.

Given a diagram of an oriented knot, we can write down a presentation of its group, called the *Wirtinger presentation*, as follows. Place one generator on each *arc* of the diagram. These are called the *Wirtinger generators*. (This has already been done in Figure 1.) Next, write down one relation for each crossing as shown in Figure 2. The orientation of the overcrossing strand is need to write down the relation! (But the orientation of the undercrossing strand is not needed.) The group of the unknot is thus $\langle x \mid \ \rangle$ which is $\mathbb{Z}$, the free group on one generator, also known as the infinite cyclic group. This group is Abelian.

58. Write down the Wirtinger presentation for the knot group for each of the knots in Figure 1. Using Tietze transformations, change the presentation of each of $3_1, 4_1, 5_1$ and 5_2 to one having two generators and one relation.

59. Start with the Wirtinger presentation for *any* knot and show that if we Abelianize the group by adding the relations that say that every generator commutes with every other generator, then we always get $\mathbb{Z}$, the free group on one generator. Explain why this means that every knot group is infinite.

60. If G is a group and $x \in G$, we say that any element of the form $g^{-1}xg$, where g is any element of G, is a *conjugate* of x. We say that $x, y \in G$ are *conjugate* if there exists $g \in G$ such that $y = g^{-1}xg$.

    (a) Show that conjugacy is an equivalence relation on the elements of G.

    (b) Given any knot diagram, show that all the Wirtinger generators are in the same conjugacy class (that is, they are all conjugate to each other).

    (c) Let $g \in G$ be fixed. Define $\Phi_g : G \to G$ by $\Phi_g(x) = g^{-1}xg$. Show that Φ is an isomorphism. (An isomorphism from a group to itself is called an *automorphism*.)

    (d) Let p be an odd prime. Find all the conjugacy classes in the dihedral group

$$D_p = \langle f, r \mid f^2 = 1, r^p = 1, frfr = 1 \rangle.$$

61. Suppose that $\sigma : G \to H$ is a group homomorphism.

    (a) Show that if x and y are conjugate elements in G, then $\sigma(x)$ and $\sigma(y)$ are conjugate elements in H.

(b) Show that if σ is an epimorphism and G is Abelian, then H is Abelian.

(c) Let X be a set of generators for G and let $Y = \{\sigma(x) \mid x \in X\} \subset H$. Show that σ is an epimorphism if and only if Y generates H.

62. The goal of this exercise is to study when a knot group has an epimorphism onto a dihedral group.

(a) Suppose that G is the group of a knot with generating set X and furthermore that all the elements of X are in the same conjugacy class. Suppose there is an epimorphism from G onto D_p for some odd prime. Use the previous exercises to conclude that all the generators of G must be sent into the same conjugacy class and determine what conjugacy class that is.

(b) Let G be the group of the trefoil 3_1 and consider the presentation with two generators and one relation that was found in Exercise 58. We want to find all possible odd primes p such that there exists an epimorphism from G onto D_p. Explain why the previous exercises imply that the two generators of the knot group must be sent to reflections, that is, to elements of the form fr^k for $0 \leq k < p$. Thus there are potentially p^2 choices for the images of the two generators. A given choice will give a homomorphism if and only if the single relator in the knot group is sent to the identity. Hence, for a given p, we have a finite number of things to check to see if there is an epimorphism onto D_p. However, we can reduce the amount of checking that must be done!

i. Suppose there is an epimorphism $\sigma : G \to D_p$. Let g be any element in D_p. Show that $\Phi_g \circ \sigma : G \to D_p$ is an epimorphism.

ii. Show that if there is an epimorphism $\sigma : G \to D_p$, then there is an epimorphism $\tau : G \to D_p$ such that τ takes one of the two generators of G to the reflection f. (Hint: We will have $\tau = \Phi_g \circ \sigma$ for some g.)

iii. Thus to see if there is an epimorphism from G to D_p, we may assume that one generator is taken to f and the other to fr^k for some k. Show that the only odd prime p for which there is an epimorphism of G onto D_p is $p = 3$.

(c) Repeat the above work for the knot 4_1 in order to find all odd primes p for which the knot group maps onto D_p.

(d) Repeat the above work for the knot 5_1 in order to find all odd primes p for which the knot group maps onto D_p.

(e) Repeat the above work for the knot 5_2 in order to find all odd primes p for which the knot group maps onto D_p.

63. Do these exercises show that the knots $0_1, 3_1, 4_1, 5_1$, and 5_2 all have different groups and hence are all different knots?

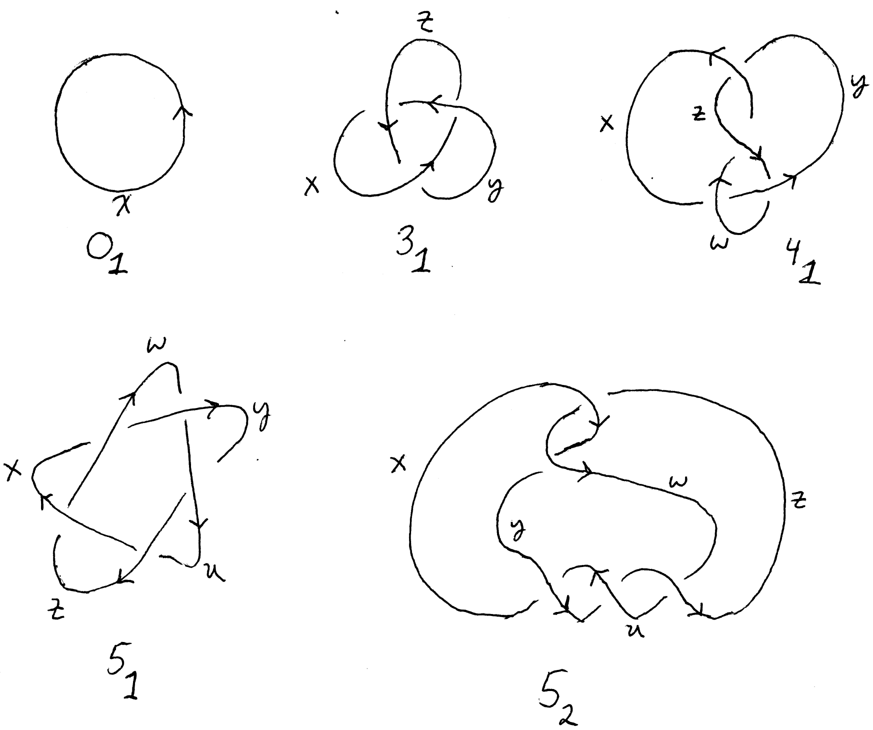
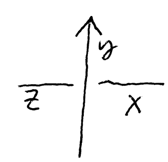


Figure 1



$$xy = yz$$

Figure 2