<u>Thm</u> Suppose N is a subgp of G.
The following are equivalent:

1) $\forall g \in G$, $gN = Ng$

2) $\forall g \in G$ $gNg^{-1} = N$

3) $\forall g \in G$ $gNg^{-1} \subset N$

4) every right coset of N is a left coset of N & viceversa

<u>Pf</u>: We'll show 4) $\Leftrightarrow$ 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3)

4) $\Rightarrow$ 1): let $g \in G$. Now $gN = Nf$ for some $f \in G$.
Now $g = g \cdot 1 \in gN = Nf$. So $g \in Nf$
but $g = 1g \in Ng$. So $Ng \cap Nf \neq \emptyset$
because $g \in Ng \cap Nf$. Therefore $Ng = Nf$
and so $gN = Ng$.

1) $\Rightarrow$ 4) trivial.

1) $\Rightarrow$ 2) Let $g \in G$. Suppose $x \in gNg^{-1}$. So $x = gng^{-1}$
for some $n \in N$. But $gN = Ng$ so $\exists$ $m \in N$ $\ni$
$gn = mg$. Thus $x = gng^{-1} = mgg^{-1} = m \in N$.
Hence $gNg^{-1} \subset N$. Now let $n \in N$. Because
$gN = Ng$, $\exists$ $m \in N$ $\ni$ $ng = gm$. Now $n = gmg^{-1} \in gNg^{-1}$
Hence $N \subset gNg^{-1}$ and so $gNg^{-1} = N$

2) ⇒ 1) Let $g \in G$. Suppose $x \in gN$.
Then $x = gn$ for some $n \in N$. Now $xg^{-1} = gng^{-1} \in gNg^{-1} = N$
So $\exists\, m \in N \ni xg^{-1} = m$ & $x = mg \in Ng$. We have shown $gN \subset Ng$. The proof that $Ng \subset gN$ is similar.
Therefore $gN = Ng$

2) ⇒ 3) trivial

3) ⇒ 2) Let $g \in G$. Since $hNh^{-1} \subset N$ $\forall\, h \in G$
Letting $h = g$ & g^{-1} we have
$gNg^{-1} \subset N$ and $g^{-1}Ng \subset N$.
Let $n \in N$. We have $n = g(g^{-1}ng)g^{-1} \in gNg^{-1}$
since $g^{-1}ng \in g^{-1}Ng \subset N$.
Thus $N \subset gNg^{-1}$. Since $N \subset gNg^{-1}$ & $gNg^{-1} \subset N$, it follows that $N = gNg^{-1}$

■

<u>Def</u> A subgroup $N \subset G$ is called <u>normal</u> if any one of the four equivalent statements in the theorem are true.