Let G be a group. If A & B are <u>subsets</u> of G (not necessarily subgroups) define AB as

$$AB = \{ab \mid a \in A \ \&\ b \in B\}.$$

A coset of a subgroup is a special case of this where one of A or B is a single element and the other is a subgroup.

<u>Lemma 1</u> multiplication of subsets is associative. i.e. If A, B, C are subsets of G then

$$(AB)C = A(BC)$$

<u>Pf</u>: Suppose $x \in (AB)C$ then $x = dc$ where $d \in AB$ & $c \in C$. Hence $d = ab$ where $a \in A$, $b \in B$ & $x = (ab)c$. But multiplication in G is associative, so

$$x = (ab)c$$
$$= a(bc) \in A(BC)$$

So $(AB)C \subset A(BC)$.

Similarly, $A(BC) \subset (AB)C$.

Thus $A(BC) = (AB)C$. $\square$

Lemma 2 If H is a subgp of G, then

$$HH = H$$

Pf: Suppose $x \in HH$. then $x = h_1 h_2$ where $h_1, h_2 \in H$. Since H is a subgp, $h_1 h_2 \in H$. Thus $HH \subset H$.
Now suppose $h \in H$. $h = 1 \cdot h \in HH$.
So $H \subset HH$. Thus $H = HH$. ∎

Thm If N is a normal subgp of G & $x, y \in G$, then $(Nx)(Ny) = Nxy$ & $(xN)(yN) = xyN$.

Pf:
$$
\begin{aligned}
(Nx)(Ny) &= N(x(Ny)) && \text{Lemma 1} \\
&= N((xN)y) && \text{Lemma 1} \\
&= N((Nx)y) && N \text{ is normal} \\
&= N(N(xy)) && \text{Lemma 1} \\
&= (NN)(xy) && \text{Lemma 1} \\
&= Nxy && \text{Lemma 2.}
\end{aligned}
$$

the proof that $(xN)(yN) = xyN$ is similar. ∎

Thus for a <u>normal subgp</u> N,

multiplying two left cosets gives another left coset, or multiply two right cosets gives another right coset. In fact, we have

<u>Theorem</u> N a subgp of G. The following are equivalent

1) N is normal

2) the product of left cosets of N is a left coset of N

3) the product of right cosets of N is a right coset of N.

<u>Pf</u> The previous theorem shows

1) $\Rightarrow$ 2) & 1) $\Rightarrow$ 3)

We'll show 2) $\Rightarrow$ 1) & 3) $\Rightarrow$ 1).

assume 3). Let $g \in G$. We want to show $gNg^{-1} \subset N$. But $gNg^{-1}N$ is the product of Right cosets and so must be a right coset. So $\exists$ $f \ni gNg^{-1}N = fN$

Now $1 \in gNg^{-1}N$ so $1 \in fN$. Hence $fN = N$ Since $fN \cap N \neq \phi$. Therefore $gNg^{-1}N = N$

Suppose $x \in gNg^{-1}$. Then $x = gng^{-1}$ for some $n \in N$. So

$$\begin{aligned} x &= gng^{-1} \\ &= gng^{-1} \cdot 1 \in gNg^{-1}N = N \end{aligned}$$

Therefore $gNg^{-1} \subset N$ and N is normal.

A similar proof shows $2) \Rightarrow 1)$

$\boxed{\text{Mal}}$

So, when N is normal, we can multiply two left cosets to obtain a left coset or two right cosets to obtain a right coset

THIS TURNS THE SET OF RIGHT, OR LEFT, COSETS OF N INTO A GROUP

<u>Thm</u> If N is a normal subgr of G, then the set of right cosets with multiplication is a group. (or same with left cosets)

<u>Pf</u> We've already seen that multiplication of subsets is associative. The group N serves as the identity:

$$(N)(xN) = (1N)(xN) = (1 \cdot x)N = xN \quad \forall x.$$

and $x^{-1}N$ is the inverse of xN : $(x^{-1}N)(xN) = x^{-1}xN = N$.

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