09.21 Chapter 14 §14.1 The Conic Sections & §14.2 (part of)

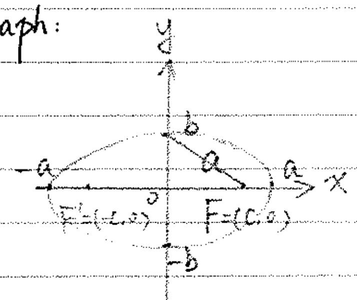
* Conic sections: ellipse, hyperbola, parabola and circle be obtained by slicing a cone with a plane

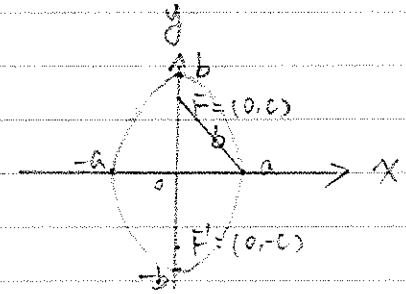
▷ Ellipse

Definition: the set of points in the plane for which the sum of the distances from two fixed points (foci) is constant.

Equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (standard form) * when $a^2 = b^2$ it is a circle

Graph:





foci on the x-axis $(\pm c, 0)$	foci on the y-axis $(0, \pm c)$
$\Rightarrow a > b$ & $a^2 = b^2 + c^2$	$\Rightarrow a < b$ & $b^2 = a^2 + c^2$
$c = \sqrt{a^2 - b^2}$	$c = \sqrt{b^2 - a^2}$
x intercepts: $(a, 0), (-a, 0)$	x, y intercepts: same
y intercepts: $(0, b), (0, -b)$	x-axis: minor axis y-axis: major axis

x axis: major axis: $\boxed{\text{the length of the long axis of the ellipse}}$

y axis: minor axis: $\boxed{\text{the length of the short axis}}$

sum distance $= 2a$	sum distance $= 2b$
Central point: $(0, 0)$	Central point: $(0, 0)$

Shifted ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ (shifted circle if $a = b$)

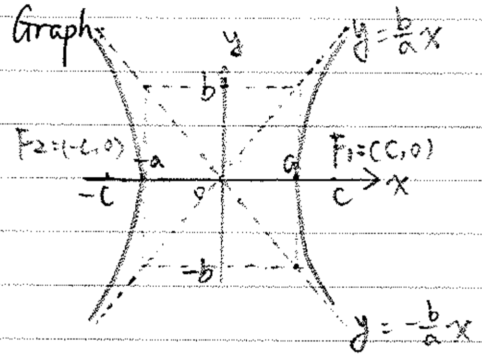
central point: (h, k)

2> Hyperbola

Definition: the set of points in the plane for which the difference of the distances from two fixed points (foci) is constant.

Equation: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ / $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

Graph:



Foci on x-axis

Equation: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$c = \sqrt{a^2+b^2}$, Foci $(\pm c, 0)$

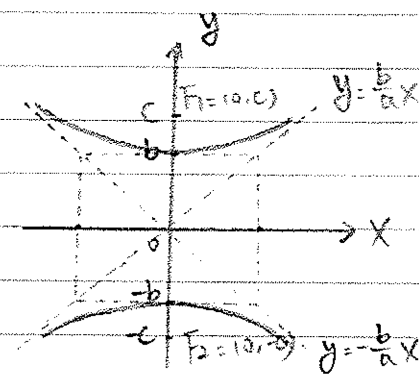
x intercepts: $(\pm a, 0)$

y intercepts: none

Asymptotes: $y = \pm \frac{b}{a}x$

difference of the distance: $2a$

Central point: $(0, 0)$



Foci on y-axis

Equation: $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

$c = \sqrt{a^2+b^2}$, Foci $(0, \pm c)$

x intercepts: none

y intercepts: $(0, \pm b)$

Asymptotes: $y = \pm \frac{b}{a}x$

difference of the distance: $2b$

Central point: $(0, 0)$

Shifted hyperbola

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

(horizontal)

Central point: (h, k)

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

(vertical)

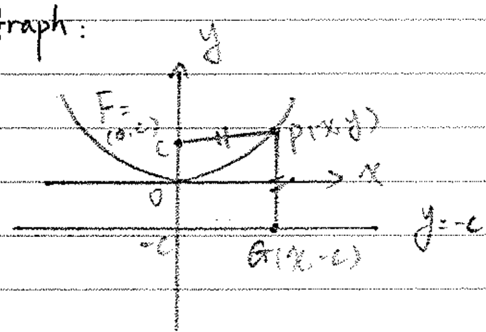
Central point: (h, k)

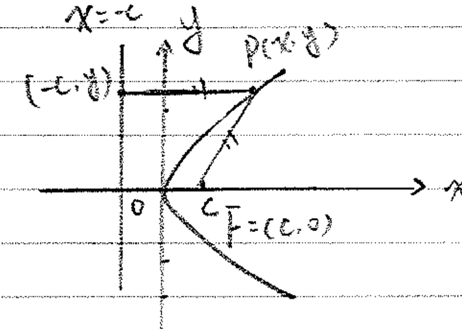
3> Parabola

Definition: the set of points in the plane for which the distances from a fixed point (focus) and a fixed line (directrix) are equal.

Equation: $y = ax^2$ / $x = by^2$

Graph:





Equation: $y = ax^2$

Focus: $(0, c)$ $a = \frac{1}{4c}$

Directrix: $y = -c$

Equation: $x = by^2$

Focus: $(c, 0)$ $b = \frac{1}{4c}$

Directrix: $x = -c$

Shifted parabola

$y - k = a(x-h)^2$ <vertical>

$x - h = b(y-k)^2$ <horizontal>

* All the curves can describe by quadratic equations in two variables.

$$Ax^2 + (Bxy) + Cy^2 + Dx + Ey + F = 0$$

Exercise: determine the type of conic.

$2x^2 + 6x + 2y^2 - 4y - 11 = 0$

$\Rightarrow 2(x^2 + 3x + \frac{9}{4}) - \frac{9}{2} + 2(y^2 - 2y + 1) - 2 - 11 = 0$

$\Rightarrow 2(x + \frac{3}{2})^2 + 2(y-1)^2 = \frac{35}{2}$

$\Rightarrow (x + \frac{3}{2})^2 + (y-1)^2 = \frac{35}{4}$ as the circle's equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ (when $a^2 = b^2$)

$\therefore$ this conic is a circle.

09.24 Chapter 14 §14.3 Functions, Graphs, and Level Surfaces

the Quiz.

1. Find the equation of the plane containing the points (1, 2, 3), (-1, 0, 3), and (-5, -1, 2)

$P.(1,2,3)$ $Q.(-1,0,3)$ $R.(-5,-1,2)$

$\overrightarrow{PQ} = (-2,-2,0)$ $\overrightarrow{QR} = (-4,-1,-1)$

$\overrightarrow{PQ} \times \overrightarrow{QR} = \begin{vmatrix} i & j & k \\ -2 & -2 & 0 \\ -4 & -1 & -1 \end{vmatrix} = 2i - 2j - 6k$ $\therefore \vec{n} = (2,-2,-6)$

$P(1,2,3)$ 6-p

Equation:

$$2(x-1) - 2(y-2) - 6(z-3) = 0$$

$$x - 1 - y + 2 - 3z + 9 = 0$$

$$x - y - 3z + 10 = 0$$

2. Describe parametrically the line of intersection between the planes $2x-3y+z=0$ and $x+y+z=1$

① $2x-3y+z=0$

② $x+y+z=1$

①−②: $x-4y=-1$

$x = 4y-1$

②×2−①: $5y+z=2$

$z = -5y+2$

$\Rightarrow$ assume $y=t$

then $\begin{cases} x = 4t-1 \\ z = -5t+2 \end{cases}$

$\therefore$ the equation is

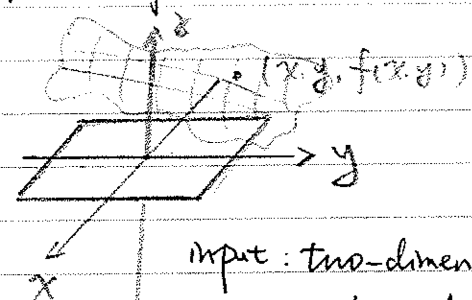
$$x = 4t-1,\ y = t,\ z = -5t+2.$$

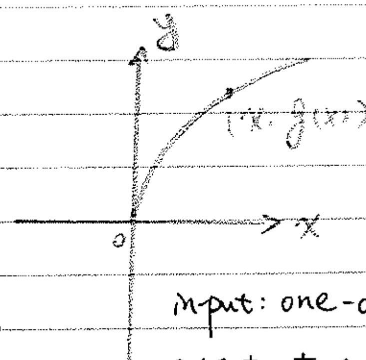
§14.3 Functions, Graphs, and Level Surfaces

$z = f(x, y)$ vs. $y = g(x)$

Ex. $z = x^2 + y^2$ $\qquad$ $y = \sqrt{x}$

Graph: Surface $\qquad$ Curve





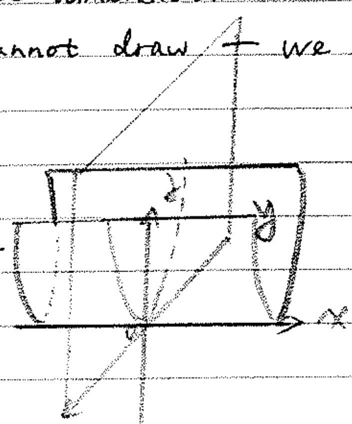
input: two-dimensional
output: three-dimensional

input: one-dimensional
output: two-dimensional

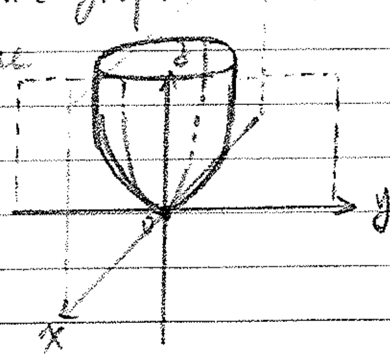
*Function of three varibles: $f(x, y, z)$
input: three-dimensional
output: cannot draw ∵ we live in three-dimensional world

Sketch graphs

Ex ①. $z = c + y^2$

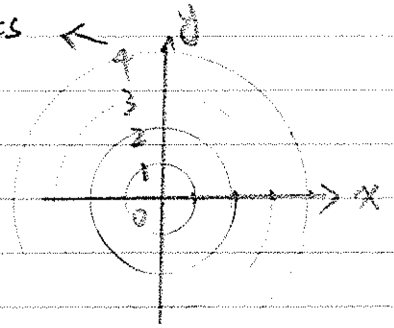


① Find that
when $x=0$, the graph is on the y-z plane, sketch the graph
② as x changes, the graph becomes a surface
sketch the surface.



Ex ② $z = x^2 + y^2$
when $x=0$, $z = y^2$
when $y=0$, $z = x^2$
when $z=0$, $x = y = 0$

Level Sets



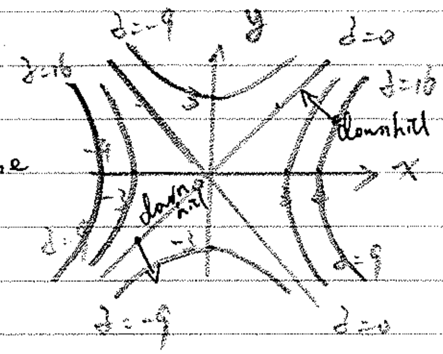
Definition:

A single level set of f is all points in the domain where the Function has a fixed value.

Ex. $f(x,y) = x^2 - y^2$

$z = x^2 - y^2$

① Put the graph in the x-y plane



1) $z=0$ $\quad y = x$ & $y = -x$

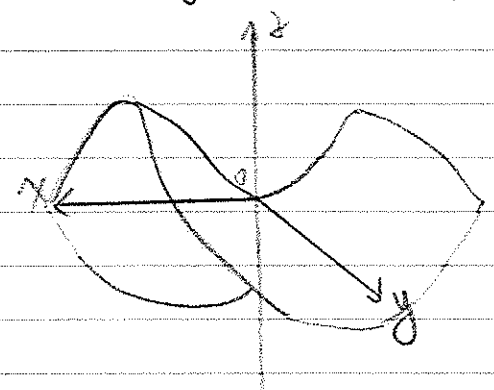
2) $z = C$

$x^2 - y^2 = C$

the graph is a hyperbola

Ex $z = 9, -9, 16$

② Sketch the graph by using the information from the level sets.



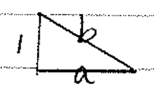
09.26

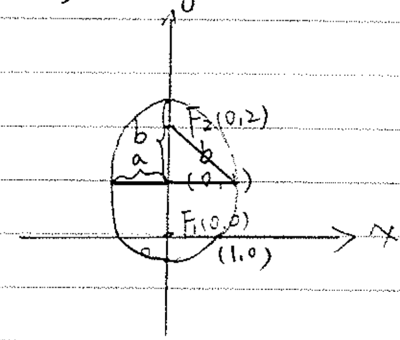
§14.2 HW 28. 29

28. Find the equation: The ellipse with x intercept $(1,0)$ and foci $(0,0)$ and $(0,2)$

Equation: $\frac{x^2}{a^2} + \frac{(y-1)^2}{b^2} = 1$

$1 = \frac{0+2}{2}$





$\Rightarrow \begin{cases} \frac{1}{a^2} + \frac{1}{b^2} = 1 & \text{— substituting } (1,0) \text{ into the equation} \\ 1^2 + a^2 = b^2 \end{cases}$

29. Find the equation of $x^2+2x+y^2-2y=2$ rotated through $\pi/3$ radians ($\frac{\pi}{3} = 60°$).

*1 conic discriminant

$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

B^2-4AC : discriminant

if $B^2-4AC<0$, then the conic is an ellipse

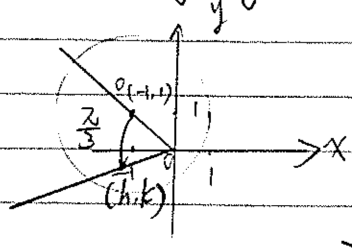
$B^2-4AC=0$, $\rightarrow$ parabola

$B^2-4AC>0$, $\rightarrow$ hyperbola

*2

if the coefficient $A=C$ & $B=0$, then the conic is a circle

$x^2+2x+y^2-2y=2 \Rightarrow (x+1)^2+(y-1)^2=4$



rotated $\pi/3 \Rightarrow$ Center(h,k), $r=2$, a circle

$(h,k) = \sqrt{2}\,(\cos\frac{13\pi}{12}, \sin\frac{13\pi}{12})$

$\Rightarrow \sqrt{2}\,(\cos(\pi+\frac{\pi}{12}), \sin(\pi+\frac{\pi}{12}))$

§14.3 HW 15, 36, 31, 34

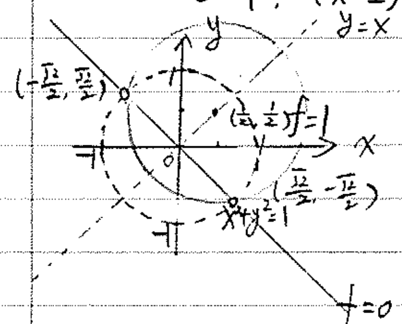
15. $f(x,y) = \frac{x+y}{x^2+y^2-1}$. Values $-2,-1,1,2$ and describe the level curve for the general value.

$f(x,y) = \frac{x+y}{x^2+y^2-1}$ domain: $x^2+y^2-1 \neq 0$.

$\Rightarrow x+y = c(x^2+y^2-1)$ level sets are always circles

when $c=0$, $y=-x$

$c=1$, $(x-\frac{1}{2})^2+(y-\frac{1}{2})^2 = \frac{3}{2}$

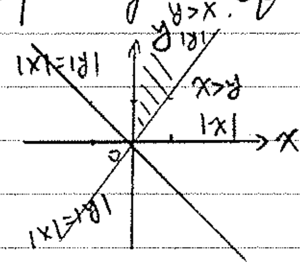


Circles with centers on the line $y=x$ & passing through the points $(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2})$. excluding points on the circle $x^2+y^2=1$.

36. sketch the surface in space defined by the equation

$z = \max(|x|, |y|)$

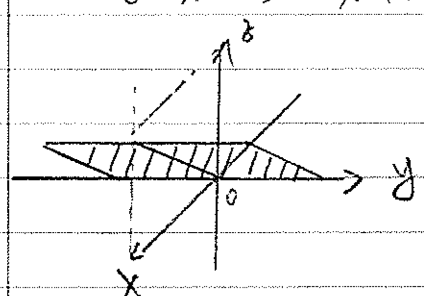
$z = \max(|x|, |y|) = \begin{cases} |x| \\ |y| \end{cases}$



sketch each part of the level sets to sketch the surface.

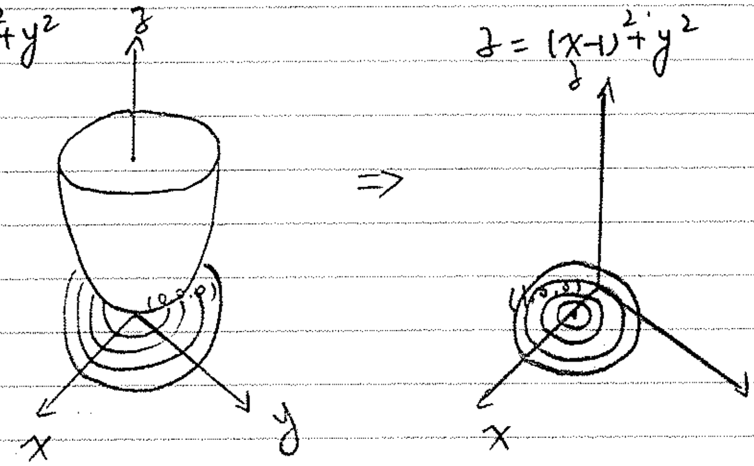
Ex. $x>y$ $|x| = x$ part

$z = x \Rightarrow -x + 0\cdot y + z = 0$ the graph is a plane $\perp (-1,0,1)$



31. Sketch the surface in space defined by $z = (x-1)^2 + y^2$

$z = x^2 + y^2$ $\qquad\qquad$ $z = (x-1)^2 + y^2$



shifted the graph of $z = x^2+y^2$ from $(0,0,0)$ to $(1,0,0)$

34. sketch the surface in space defined by

$z = 3x^2 + 3y^2 - 6x + 12y + 15$

the level sets are circles, same way as Q31.

Rotation of Axes

* 44(c) $\quad z = x^2 + xy$.

$\Rightarrow \quad x^2 + xy - c = 0$. $\qquad \because B^2 - 4AC = 1 - 4 \times 1 \times 0 > 0$

$\therefore$ the conic is a hyperbola

* Equation of the rotated α

$$\alpha = \frac{1}{2}\tan^{-1}\left[\frac{B}{A-C}\right]$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{B}{A-C}\right) = \frac{1}{2}\tan^{-1}\left(\frac{1}{1-0}\right) = \frac{\pi}{8}$$

$$\Rightarrow \begin{cases} x = (\cos\frac{\pi}{8})\,\overline{X} - (\sin\frac{\pi}{8})\,\overline{Y} \\ y = (\sin\frac{\pi}{8})\,\overline{X} + (\cos\frac{\pi}{8})\,\overline{Y} \end{cases}$$

Substituting into $x^2 + xy - c = 0$

We can get the equation of a hyperbola by using $\overline{X}, \overline{Y}$