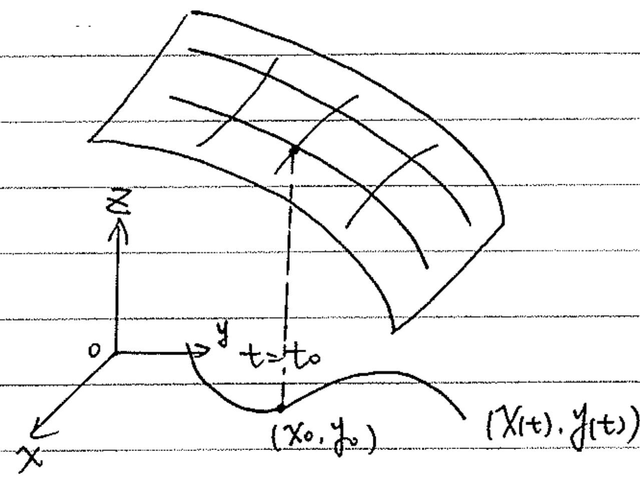
10.19 Chapter 15 §15.4 Matrix Multiplication and the Chain Rule

$z = f(x, y)$

$x(t) \qquad \Delta x \approx x'(t_0)\cdot \Delta t$

$y(t) \qquad \Delta y \approx y'(t_0)\cdot \Delta t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$



$$f(x+\Delta x, y+\Delta y) \approx T(x+\Delta x, y+\Delta y) = f(x_0, y_0) + \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}\Delta x + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}\Delta y$$

$$\Delta f = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) \approx \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}\Delta x + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}\Delta y$$

$$\Delta f = \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}\cdot x'(t_0)\cdot\Delta t + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)} y'(t_0)\cdot\Delta t$$

$$\frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}\cdot x'(t_0) + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}\cdot y'(t_0)$$

$$\Rightarrow \frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \quad \text{(Chain Rule)}$$

Matrix Multiplication

$$A^{n\times m}\, B^{m\times \ell} = C^{n\times \ell}$$

* the entry in the i-th row and j-th column of C is the dot product of the i-th row of A with j-th column of B

Ex 1

$$\underset{2\times3}{\begin{pmatrix}1 & 2 & 3\\ 0 & -1 & 4\end{pmatrix}}\underset{3\times1}{\begin{pmatrix}4\\5\\6\end{pmatrix}} = \underset{2\times1}{\begin{pmatrix}1\times4+2\times5+3\times6\\ 0\times4-1\times5+4\times6\end{pmatrix}} = \begin{pmatrix}32\\19\end{pmatrix}$$

Ex 2

$$\underset{2\times3}{\begin{pmatrix}1 & 2 & 3\\ 0 & -1 & 4\end{pmatrix}}\underset{3\times2}{\begin{pmatrix}4 & 0\\5 & 0\\6 & 1\end{pmatrix}} = \underset{2\times2}{\begin{pmatrix}1\times4+2\times5+3\times6 & 1\times0+2\times0+3\times1\\ 0\times4-1\times5+4\times6 & 0\times0-1\times0+4\times1\end{pmatrix}}$$

$$= \begin{pmatrix}32 & 3\\ 19 & 4\end{pmatrix}$$

$$A(B+C) = AB + AC$$

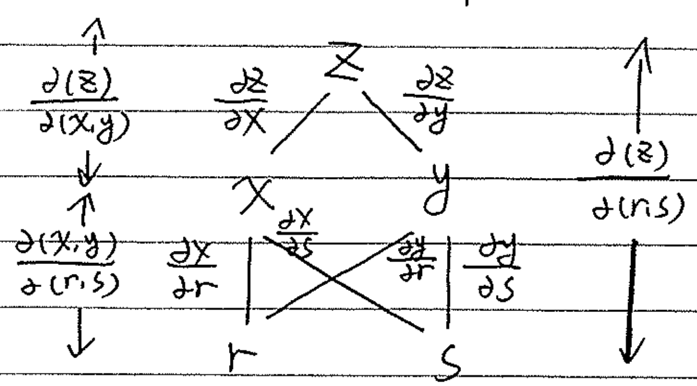
* The order will make a difference.

$$\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}-1 & 0\\2 & 3\end{pmatrix} = \begin{pmatrix}1\times(-1)+1\times2 & 1\times0+1\times3\\ 1\times(-1)+0\times2 & 1\times0+0\times3\end{pmatrix} = \begin{pmatrix}1 & 3\\-1 & 0\end{pmatrix}$$

$$\begin{pmatrix}-1 & 0\\2 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}-1\times1+0\times1 & -1\times1+0\times0\\ 2\times1+3\times1 & 2\times1+3\times0\end{pmatrix} = \begin{pmatrix}-1 & -1\\5 & 2\end{pmatrix}$$

$$\begin{pmatrix}1 & 3\\-1 & 0\end{pmatrix} \neq \begin{pmatrix}-1 & -1\\5 & 2\end{pmatrix}$$

Combine Matrix Multiplication & Chain Rule



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$ → looks like a fraction but cannot be cancelled

$$\overset{1\times 2}{\begin{pmatrix} \frac{\partial z}{\partial r} & \frac{\partial z}{\partial s} \end{pmatrix}} = \overset{1\times 2}{\begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix}} \overset{2\times 2}{\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{pmatrix}}$$

$$= \overset{1\times 2}{\begin{pmatrix} \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} & \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \end{pmatrix}}$$

$$\Rightarrow \quad \frac{\partial(z)}{\partial(r,s)} = \frac{\partial(z)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)}$$

Suppose $u_1, u_2 \cdots u_n$ are each functions of $x_1 . x_2 \cdots x_m$

The $n \times m$ matrix

$$\frac{\partial(u_1, u_2 \cdots u_n)}{\partial(x_1, x_2 \cdots x_m)} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_m} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & & \frac{\partial u_n}{\partial x_m} \end{pmatrix}$$

Ex. $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$

$$\left(\frac{dz}{dx}\right) = \left(\frac{dz}{dy}\right)\left(\frac{dy}{dx}\right)$$

1×1 matrix

$$= \left(\frac{dz}{dy} \cdot \frac{dy}{dx}\right)$$

HW §15.3 (13) (15)

13. Describe the collection of vectors tangent to all possible curves on the paraboloid $z=x^2+y^2$ through the point $(1,2,5)$

$\Rightarrow$ tangent plane

$\vec{n}\ (-f_x, -f_y, 1)$ $\quad$ $f_x = 2x = 2$

$f_y = 2y = 4$

$\vec{v}\cdot\vec{n}=0$

Assume $\vec{v} = (a.b.c)$, $\vec{n}\ (-2,-4,1)$

$$\underline{-2a-4b+c=0}$$

Vectors are (ai, bj, ck) where $-2a-4b+c=0$

15. (a) Use the chain rule to find $(d/dx)(x^x)$ by using the function $f(y,z)=y^z$

$y=x,\ z=x$

$$\frac{d}{dx} = \frac{\partial f}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dx}$$

$$= z\cdot y^{z-1} + y^z\cdot \ln z = y^z(\ln z+1) = x^x(\ln x+1)$$

(b) Calculate $(d/dx)(x^x)$ by using one-variable calculus

$f(x) = x^x$

$$\frac{d}{dx}\cdot x^x = x^x\cdot\ln x + x\cdot x^{x-1}$$

$$= x^x\cdot\ln x + x^x$$

$$= x^x(\ln x+1)$$

10.24 Chapter 16 §16.1 Gradients and Directional Derivatives

§15.3 HW

16. $T(x, y, z) = x^2 + y^2 + z^2$

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases}$$

1) What is $T'(t)$?

<1> $T(\cos t, \sin t, t) = \cos^2 t + \sin^2 t + t^2$

$$= 1 + t^2$$

$$\frac{dT}{dt} = 2t$$

<2> Chain Rule

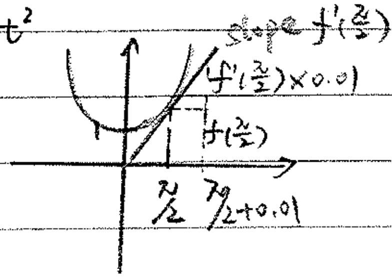
$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial T}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial T}{\partial z}\cdot\frac{dz}{dt}$$

$$= \cancel{-2x\cdot\sin t + 2y\cdot\cos t + 2t\cdot 1}$$

$$= 2t$$

2) Find the approximate value at $t = (\pi/2) + 0.01$

$T = 1 + t^2$

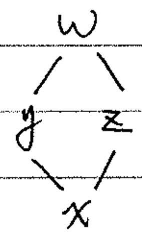


$$T\left(\frac{\pi}{2} + 0.01\right) \approx T\left(\frac{\pi}{2}\right) + T'\left(\frac{\pi}{2}\right)\times 0.01$$

$$= 1 + \frac{\pi^2}{4} + \frac{\pi}{100}$$

15. a) Use the chain rule to find $(d/dx)(x^x)$ by using the function $f(y,z) = y^z$

$f(y,z) = w$



$w = y^z$

$y = x$

$z = x$

$$\frac{dw}{dx} = \frac{\partial w}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dx}$$

$$= z \cdot y^{z-1} + y^z \cdot \ln y$$

$$= x \cdot x^{x-1} + x^x \cdot \ln x$$

$$= x^x (1 + \ln x)$$

b) Calculate $(d/dx)(x^x)$ by using one-variable calculus

$$\frac{dw}{dx} = \frac{dx^x}{dx} = \frac{d(e^{\ln x})^x}{dx} = \frac{d\,e^{x \ln x}}{dx} = e^{x \ln x} \cdot (\ln x + x \cdot \tfrac{1}{x})$$

$$= x^x (\ln x + 1)$$

§16.1 Gradients and Directional Derivatives

* Suppose f is a function of many variables $x_1, x_2 \cdots x_n$, the gradient of f, written ∇f ("nabla") is:

$$\nabla f(x_1, x_2 \dots x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots \frac{\partial f}{\partial x_n}\right)$$

Ex. $f(x,y,z) = x^2 + e^{yz}$

$$\nabla f(x,y,z) = (2x,\ z \cdot e^{yz},\ y \cdot e^{yz})$$

* Suppose $\vec{u} = (u_1, u_2 \cdots u_n)$ is a unit vector in the domain of $f(x_1, x_2 \cdots x_n)$, then the directional derivative of f at $(x_1^\circ, x_2^\circ, \cdots x_n^\circ)$ is $\nabla f(x_1^\circ, x_2^\circ \cdots x_n^\circ) \cdot \vec{u}$

Ex. $f(x,y,z) = x^2+y^2+z^2$

$\nabla f(x,y,z) = (2x, 2y, 2z)$

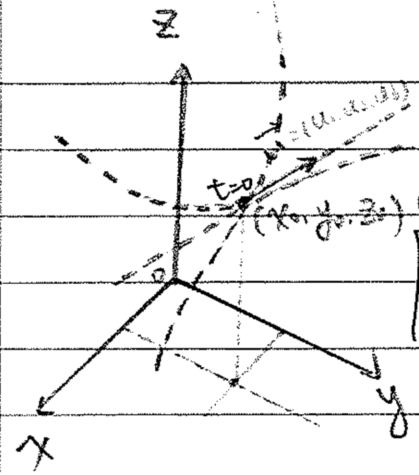
Let $\vec{u} = \dfrac{(1,2,3)}{\sqrt{14}}$, the directional derivative of f at $(-1,0,5)$ in the direction of $\vec{u}$ is:

$$\nabla f(-1,0,5) \cdot \vec{u}$$
$$= 2(-1,0,5) \cdot \frac{(1,2,3)}{\sqrt{14}}$$
$$= \frac{-2+30}{\sqrt{14}} = 2\sqrt{14}$$

Ex2. if $\vec{u} = (1,0,0)$

the directional derivative of f at (x_0, y_0, z_0) is

$$\nabla f(x_0, y_0, z_0) \cdot \vec{u}$$
$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\Bigg|_{(x_0,y_0,z_0)} \cdot (1,0,0)$$
$$= \frac{\partial f}{\partial x}\Bigg|_{(x_0,y_0,z_0)}$$



$$\begin{cases} x = x_0 + t \\ y = y_0 + t \\ z = z_0 + t \end{cases} \Rightarrow \quad \frac{dx}{dt} = u_1, \quad \frac{dy}{dt} = u_2, \quad \frac{dz}{dt} = u_3$$

In general,

$$\left.\frac{df}{dt}\right|_{t=0} = \left.\left(\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}\right)\right|_{t=0}$$

$$= \left.\left(\frac{\partial f}{\partial x}\cdot u_1 + \frac{\partial f}{\partial y}\cdot u_2 + \frac{\partial f}{\partial z}\cdot u_3\right)\right|_{t=0}$$

$$= \left(\frac{\partial f}{\partial x}(x_0, y_0, z_0), \frac{\partial f}{\partial y}(x_0, y_0, z_0), \frac{\partial f}{\partial z}(x_0, y_0, z_0)\right)\cdot(u_1, u_2, u_3)$$

$$= \nabla f(x_0, y_0, z_0)\cdot\vec{u}$$

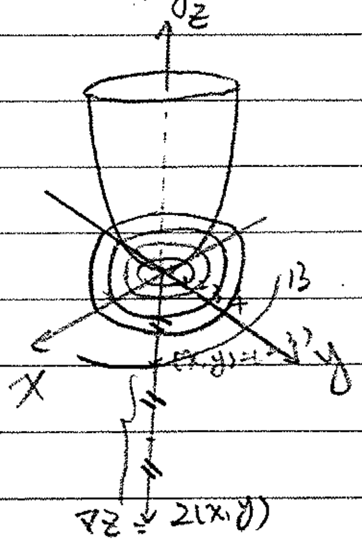
* $\nabla f \cdot \vec{u} = \|\nabla f\| \cdot \|\vec{u}\| \cdot \cos\theta$

$= \|\nabla f\| \cdot \cos\theta$ ($\vec{u}$ is a unit vector)

$\Rightarrow \quad -\|\nabla f\| \le \nabla f \cdot \vec{u} \le \|\nabla f\|$

* the gradient 1) is a vector; 2) in the domain of the function; 3) it's always perpendicular to level-sets

Ex. $z = x^2 + y^2$



$\nabla z = (2x, 2y)$

$= 2(x, y)$

$(x, y) = (2, 3)$

$\nabla z = (4, 6)$

* all gradient vectors define a vector field