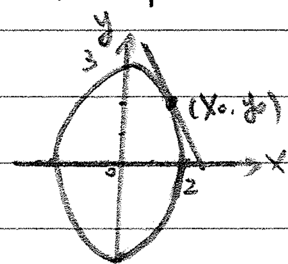
10.26 Chapter 16 §16.2 Gradients, Level Surfaces and Implicit Differentiation

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$



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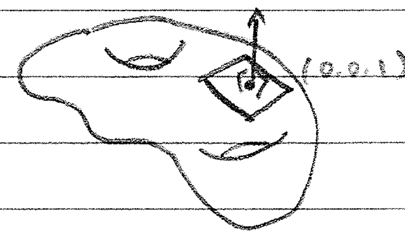


Find the line tangent to the ellipse at (x_0, y_0)

Find the plane tangent to the surface $5e^{x-y}+\sin(xz)=5$ at $(0.0.1)$

Let $f(x,y,z)=5e^{x-y}+\sin(xz)$

$$\nabla f=(5e^{x-y}+z\cdot\cos(xz),\ -5e^{x-y},\ x\cos(xz))$$

Assume $(0.0.1)$ is on the level set $f=5$

($\because$ gradient is perpendicular to the level set. We can get the equation of the tangent plane by getting a normal vector [gradient] and a point on the plane [(0.0.1)])

$$\nabla f(0.0.1)=(6,-5,0)$$

Equation: $6(x-0)-5(y-0)+0\times(z-1)=0$

$$\Rightarrow 6x-5y=0$$

Implicit Differentiation

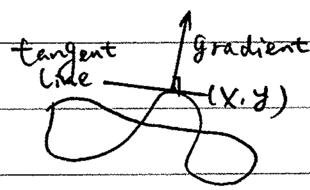
Ex. §, 6.2 Exercises 28

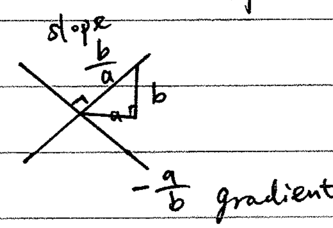
$y - \sin(x^3) + x^2 - y^2 = 1$. find dy/dx.

1> $\frac{dy}{dx} - \cos(x^3) \times 3x^2 + 2x - 2y \cdot \frac{dy}{dx} = 0$.

$(1-2y)\frac{dy}{dx} = 3x^2\cos(x^3) - 2x$

$\frac{dy}{dx} = \frac{3x^2\cos(x^3) - 2x}{1-2y}$ <1> Implicit Differentiation

2> 



$f(x,y) = y - \sin(x^3) + x^2 - y^2$

level set $f = 1$ <2> Gradient and Level set

$\nabla f = (-3x^2\cos(x^3) + 2x,\ 1-2y)$

Assume $a = -(1-2y)$, $b = -3x^2\cos(x^3) + 2x$

$\frac{dy}{dx} = \frac{b}{a} = \frac{-3x^2\cos(x^3)+2x}{-(1-2y)} = \frac{3x^2\cos(x^3)-2x}{1-2y}$

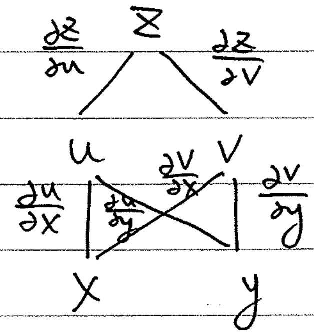
Same as the result of 1>

$\Rightarrow \quad \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$* \quad \frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

HW §15.4 21, 42

21. $z = u^2 + v^2$, $\quad u = 2x+7$, $\quad v = 3x+y+7$, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.



$$\underset{1\times 2}{\frac{\partial(z)}{\partial(x,y)}} = \underset{1\times 2}{\frac{\partial(z)}{\partial(u,v)}} \cdot \underset{2\times 2}{\frac{\partial(u,v)}{\partial(x,y)}}$$

$$\begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

$$= \begin{pmatrix} 2u & 2v \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4u+6v & 2v \end{pmatrix}$$

$$\frac{\partial z}{\partial x} = 4u+6v = 4(2x+7)+6(3x+y+7)$$
$$= 26x+6y+70$$

$$\frac{\partial z}{\partial y} = 2v = 6x+2y+14$$

42. a) $\left| \frac{\partial(u,v)}{\partial(x,y)} \; \frac{\partial(x,y)}{\partial(t,s)} \right| = \left| \frac{\partial(u,v)}{\partial(t,s)} \right|$

$\Rightarrow \; |A \; B| = |A| \cdot |B|$

b) $\frac{\partial(x,y)}{\partial(t,s)} \cdot \frac{\partial(t,s)}{\partial(x,y)} \underset{\uparrow}{=} \frac{\partial(x,y)}{\partial(x,y)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

Chain Rule

Supplement

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \cdots \quad \underset{\underset{I_n}{\parallel}}{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}} \Rightarrow \begin{array}{l} I_n A^{n\times n} = A \\ A \cdot I_n = A \end{array}$$

$n \cdot 1 = 1 \cdot n = n$ for all n multiplicative identity

$n + 0 = 0 + n = n$ for all n. additive identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$\Rightarrow \begin{cases} n + (-n) = 0 \\ n \cdot \frac{1}{n} = 1 \end{cases}$$ if $n \neq 0$, there is m such that $n \cdot m = 1$

$$|A| \cdot |B| = \left|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 1$$

if $n = 0$ $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

formula

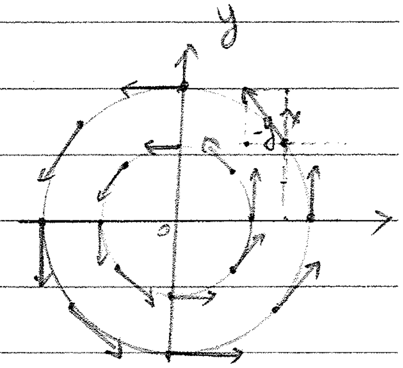
if $ad - bc \neq 0$.

10.29 Chapter 16 §16.3 Maxima and Minima

HW §16.1

11. show that $\Phi(x,y) = x\hat{j} - y\hat{i}$ is not a gradient vector field.

$\Phi(x,y) = (-y, x) \stackrel{?}{=} \nabla f$



if $\Phi(x,y) = \nabla f$

$$(-y, x) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

~~by using the mixed partial derivatives~~

$$\frac{\partial^2 f}{\partial x \partial y} = -1, \quad \frac{\partial^2 f}{\partial y \partial x} = 1$$

$$\because \frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$$

the law of equality of mixed partial derivatives doesn't hold

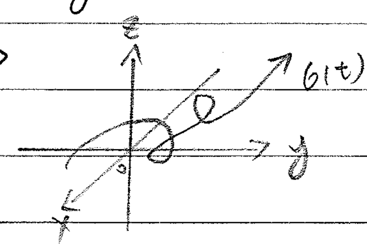
$$\therefore \Phi(x,y) \neq \nabla f$$

$\Phi(x,y)$ is not a gradient vector field.

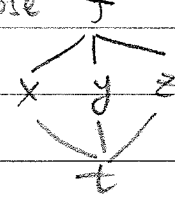
5. $f(x,y) = \ln\sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} \cdot \cancel{\tfrac{1}{2}}(x^2+y^2)^{-\frac{1}{2}} \cdot \cancel{2}x \cdots$$

19. Suppose that $f(\sigma(t))$ is an increasing function of t. What can you say about the angle between the gradient ∇f and the velocity vector σ'?



Suppose



$\because f(\sigma(t))$ is an increasing function

$$\frac{df}{dt} = f'(\sigma(t)) > 0$$

$$\because \frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\cdot\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$= \nabla f \cdot \sigma'(t)$$

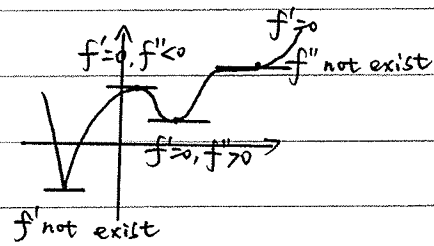
$$= \|\nabla f\| \cdot \|\sigma'(t)\| \cdot \cos\theta$$

$$\|\nabla f\|, \|\sigma'(t)\| > 0$$

$$\therefore \cos\theta > 0$$

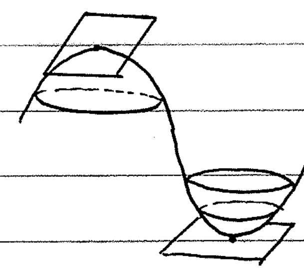
$$\therefore \theta \in \left(0, \frac{\pi}{2}\right)$$

§16.3 Maxima and Minima.



V.S.

$\Rightarrow$



one variable

$\Rightarrow \nabla f = (f')$.

If f is differentiable then the tangent plane must be horizontal

Maxima/Minima may exist when
- $f' = 0$
- f' doesn't exist

Maxima/Minima may exist when
- tangent plane is horizontal
- tangent plane doesn't exist.

A critical point of a function $f(x)$ is a point where $f' = 0$ / f' doesn't exist

* a critical point of a function, $f(x_1, x_2 \dots x_n)$ is a point where either

1) $\nabla f = 0$

or 2) ∇f doesn't exist.

Ex. find the critical points of $f(x,y) = x^2 - 2y^2$

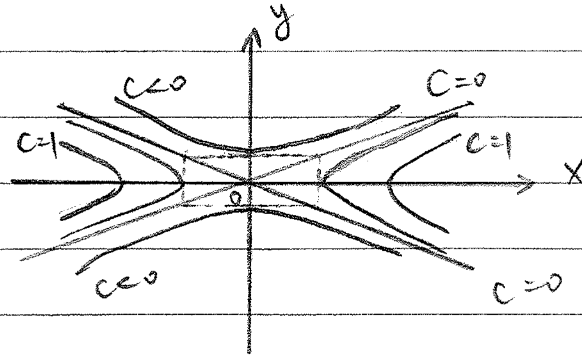
$\nabla f = (2x, -4y)$

$\nabla f = 0$ if and only if $\begin{cases} x = 0 \\ y = 0 \end{cases}$

$\therefore$ $(0,0)$ is the only critical point of $f(x,y) = x^2 - 2y^2$

Level sets of $f(x,y) = x^2 - 2y^2$

$$\begin{cases} x^2 - 2y^2 = 0 \Rightarrow y = \pm \frac{\sqrt{2}}{2} x \\ x^2 - 2y^2 = c,\ c \neq 0 \Rightarrow \frac{x^2}{c} - \frac{y^2}{\frac{c}{2}} = 1 \end{cases}$$



(0,0) is a saddle point
neither a maxima
nor a minima
of $f(x,y) = x^2 - 2y^2$

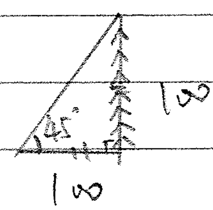
We can also know that by using the second derivative test.

First Derivative Test → critical point or not

Second Derivative Test → Local maximum / minimum / Saddle point

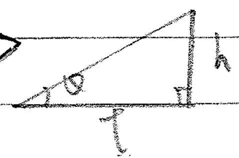
Quiz Question

1>



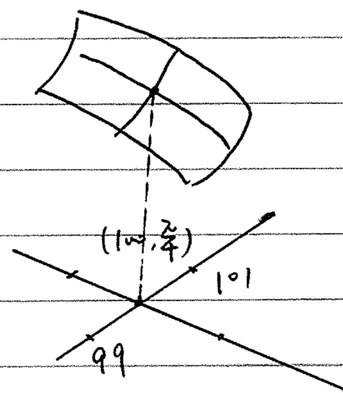
100 feet

2>



$h = l \cdot \tan\theta$

3>

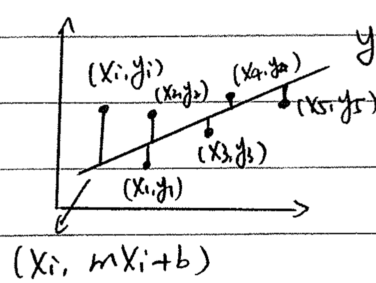


$$\begin{cases} \dfrac{\partial h}{\partial l} = \tan\theta \\ \dfrac{\partial h}{\partial \theta} = l \cdot \sec^2\theta \end{cases}$$

$$\Rightarrow \Delta h \approx \left.\frac{\partial h}{\partial l}\right|_{(100, \frac{\pi}{4})} \cdot \Delta l + \left.\frac{\partial h}{\partial \theta}\right|_{(100, \frac{\pi}{4})} \cdot \Delta\theta$$

Where $\begin{cases} \Delta l = \pm 1 \\ \Delta\theta = \pm \frac{5}{180}\pi \end{cases}$

Find the line in scatter diagram.



$$E(m, b) = \sum_{i=1}^{n} (y_i - m x_i - b)^2$$

$\downarrow$

minimum of $E(m, b)$

$\downarrow$

$\frac{\partial E}{\partial m}, \frac{\partial E}{\partial b} = 0.$

10.31 § 16.1 HW 33, 35, 43, 45, 41

HW 33

$T(x,y,z) = e^{-x^2-2y^2-3z^2}$, $P(1,1,1)$, T: temperature of ship given the location (x,y,z)

(a) in what direction T decrease most rapidly?

(b) if the ship travels at e^8 meters per second. how fast will be the temperature decrease if she proceeds in that direction?

(c) if cooled at a rate no more than $\sqrt{14}e^2$ degrees per second. Describe the set of possible directions.

a). $T(x,y,z) = e^{-x^2-2y^2-3z^2} = \dfrac{1}{e^{x^2+2y^2+3z^2}}$

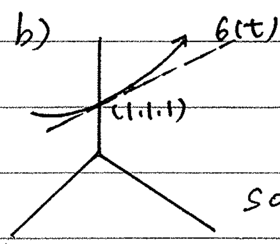
$\nabla T = \left(-2x e^{-x^2-2y^2-3z^2}, -4y e^{-x^2-2y^2-3z^2}, -6z e^{-x^2-2y^2-3z^2}\right)$

$= -2T(x, 2y, 3z)$

$\nabla T(1,1,1) = -2e^{-6}(1,2,3)$

direction $\downarrow_{max} = -\nabla T = 2e^{-6}(1,2,3)$.

b)



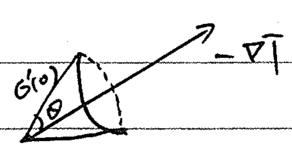
$\sigma(t) = (1,1,1) + t\dfrac{(1,2,3)}{\sqrt{14}} \cdot e^8$

$\sigma'(t) = \dfrac{(1,2,3)}{\sqrt{14}} \cdot e^8$

so that speed $= |\sigma'(t)| = e^8$

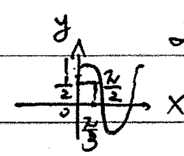
$\nabla T(1,1,1) \cdot \sigma'(0) = -2e^{-6}(1,2,3) \cdot \dfrac{(1,2,3)}{\sqrt{14}} \cdot e^8 = -2\sqrt{14}e^2$

c)



$\nabla T(1,1,1) \cdot \sigma'(0) = \|\nabla T(1,1,1)\| \cdot \|\sigma'(0)\| \cdot \cos\theta$

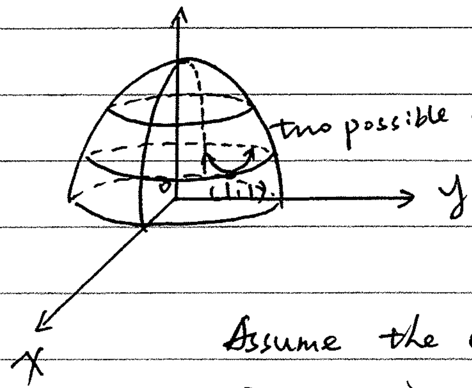
$= 2\sqrt{14}e^2 \cdot \cos\theta$



$2\sqrt{14}e^2 \cdot \cos\theta \le e^2 \cdot \sqrt{14}$

$\cos\theta \le \frac{1}{2} \Rightarrow \theta \ge \frac{\pi}{3}$

35. $z = c - ax^2 - by^2$, $P(1,1)$, find the direction has an angle whose tangent is 0.03.



$z = c - ax^2 - by^2$

$\nabla z = (-2ax, -2by)$

$\nabla z(1,1) = (-2a, -2b)$

Assume the direction is $\vec{u} = (u_1, u_2)$, $u_1^2 + u_2^2 = 1$

$\nabla z(1,1) \cdot \vec{u} = 0.03$

$$\begin{cases} -2au_1 - 2bu_2 = 0.03 & ① \\ u_1^2 + u_2^2 = 1 & ② \end{cases}$$

43. a) in what direction is the directional derivative of $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ at $(1,1)$ equal to zero?

$\nabla f = \frac{4xy}{(x^2+y^2)^2} \cdot (y, -x)$

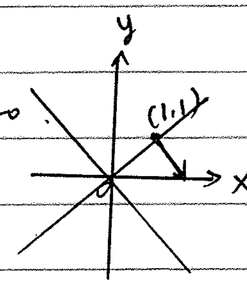
$\nabla f(1,1) = (1, -1)$

$\nabla f \cdot \vec{u} = 0$

$f(1,1) = 0$

level set $f = 0$

$x = \pm y$



Assume $\vec{u} = (u_1, u_2)$ & $u_1^2 + u_2^2 = 1$

$\Rightarrow u_1 = u_2 = \frac{\sqrt{2}}{2}$

$\vec{u}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

b) $\nabla f = (y_0, -x_0)$

$u_1 y_0 = u_2 x_0$ & $u_1^2 + u_2^2 = 1 \Rightarrow \begin{matrix} u_1 = x_0 \\ u_2 = y_0 \end{matrix}$

$\vec{u} = \frac{(x_0, y_0)}{\sqrt{x_0^2 + y_0^2}}$

c) describe the level curves of f. in terms of the result of (b)

$x>0,\ y>0$

$f=c$ if $c=0$. $y=x$

if $c\neq 0$ $\quad \dfrac{x^2-y^2}{x^2+y^2}=c$

$(1-c)x^2=(c+1)y^2$

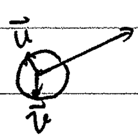
$y=\pm\sqrt{\dfrac{1-c}{1+c}}\,x \quad (c\neq -1)$ $\rightarrow$ half lines emanating from the origin

if $c=-1$

$x=0$ (y-axis)

45. $f(x,y)$ at $(1,3)$ $\begin{cases} \text{has a directional derivative } +2 \\ \text{in the direction toward to } (2,3) \\ \text{has } -2 \\ \text{in the direction toward to } (1,4) \end{cases}$

Determine the gradient vector at $(1,3)$ & compute the directional derivative in the direction toward $(3,6)$



$\nabla f=(a,b)$

$\begin{cases} \vec{u} \ \& \ (a,b)\cdot\vec{u} \\ \vec{v} \ \& \ (a,b)\cdot\vec{v} \end{cases}$

$\vec{u}=(2,3)-(1,3)=(1,0)$

$\vec{v}=(1,4)-(1,3)=(0,1)$

$(a,b)\cdot(1,0)=a=2$

$(a,b)\cdot(0,1)=b=-2$

$\nabla f(1,3)=(2,-2)$

$\vec{w}=(3,6)-(1,3)=(2,3)$

$\nabla f(1,3)\cdot\dfrac{\vec{w}}{\sqrt{13}}=\dfrac{(2,-2)\cdot(2,3)}{\sqrt{13}}=-\dfrac{2}{\sqrt{13}}$

41. Show that: $\nabla(fg) = f\cdot\nabla g + g\cdot\nabla f$

1) suppose $f(x_1, x_2 \dots x_n)$

$g(x_1, x_2 \dots x_n)$

$$\nabla(fg) = \left(\frac{\partial fg}{\partial x_1}, \frac{\partial fg}{\partial x_2} \dots \frac{\partial fg}{\partial x_n}\right)$$

$$= \left(f\cdot\frac{\partial g}{\partial x_1} + g\cdot\frac{\partial f}{\partial x_1}, f\cdot\frac{\partial g}{\partial x_2} + g\cdot\frac{\partial f}{\partial x_2}, \dots f\cdot\frac{\partial g}{\partial x_n} + g\cdot\frac{\partial f}{\partial x_n}\right)$$

$$= \left(f\frac{\partial g}{\partial x_1}, f\frac{\partial g}{\partial x_2}, \dots f\frac{\partial g}{\partial x_n}\right) + \left(g\cdot\frac{\partial f}{\partial x_1}, g\cdot\frac{\partial f}{\partial x_2}, \dots g\cdot\frac{\partial f}{\partial x_n}\right)$$

$$= f\cdot\nabla g + g\cdot\nabla f$$

2) Assume $f = mx + b$

$g = nx + c$

$$fg = (mx+b)\cdot(nx+c)$$

$$= mnx^2 + (mc+bn)x + bc$$

$$\nabla fg = \frac{dfg}{dx} = 2mnx + mc + bn = A$$

$$f\cdot\nabla g + g\cdot\nabla f$$

$$= (mx+b)\cdot n + (nx+c)\cdot m$$

$$= 2mnx + mc + bn = A$$