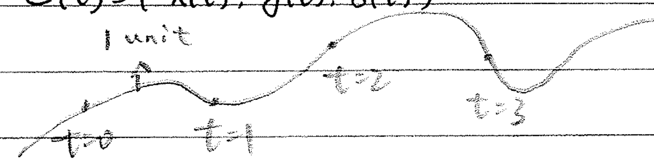
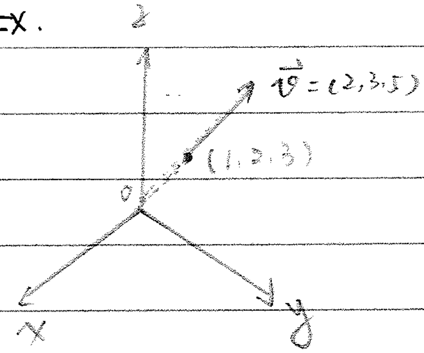
10.05 Revision

1> Parametrization by Arc Length

$\sigma(t) = (x(t), y(t), z(t))$



Ex.



Equation of the line:

$(1,2,3) + t\,(2,3,5)$

$\sigma(t) = (1+2t,\ 2+3t,\ 3+5t)$

$\sigma'(t) = (2,3,5)$

$\|\sigma'(t)\| = \sqrt{4+9+25} = \sqrt{38}$

Find the vector of the line that has unit speed.

$\tilde{L}(t) = (1,2,3) + t \cdot \frac{\vec{v}}{\|\vec{v}\|}$

$\quad = (1,2,3) + t \cdot \frac{(2,3,5)}{\sqrt{38}}$

$\tilde{L}'(t) = \frac{\vec{v}}{\|\vec{v}\|}$

$\|\tilde{L}'(t)\| = \left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = 1$.

2> Curvature

* unit tangent vector $T(t) = \frac{\sigma'(t)}{\|\sigma'(t)\|}$ / $T(s) = \frac{\sigma'(s)}{\|\sigma'(s)\|}$

$\Rightarrow T \cdot T = 1 \Rightarrow T \cdot \frac{dT}{ds} + \frac{dT}{ds} \cdot T = 0 \Rightarrow T \cdot \frac{dT}{ds} = 0$

(derivative both sides)

$k = \| T'(s) \| = \| \frac{dT}{ds} \| \rightarrow$ curvature

Principal normal vector $N = \dfrac{\frac{dT}{ds}}{\| \frac{dT}{ds} \|}$.

Ex.

Suppose a particle is moving with constant speed.

the curve is $\sigma(t)$

velocity $= \sigma'(t)$

Constant speed $= \| \sigma'(t) \| = \sqrt{\sigma'(t) \cdot \sigma'(t)} \Rightarrow$ constant $= \sigma'(t) \cdot \sigma'(t)$

$\rightarrow$ (derivative both sides) $\quad 0 = \sigma''(t) \cdot \sigma'(t) + \sigma'(t) \cdot \sigma''(t)$

$\Rightarrow \quad \sigma'(t) \cdot \sigma''(t) = 0$

$\Rightarrow \quad \sigma'(t) / \vec{v} \perp \sigma''(t) / \vec{a}$

HW 15. show that a circle of radius r has constant curvature $1/r$

1) $\sigma(t) = (r_0 \cdot \cos(ct), r_0 \cdot \sin(ct))$

$\sigma'(t) = (-r_0 c \cdot \sin(ct), r_0 \cdot c \cdot \cos(ct))$

$\| \sigma'(t) \| = \sqrt{r_0^2 c^2 \sin^2(ct) + r_0^2 c^2 \cdot \cos^2(ct)}$

$= \sqrt{r_0^2 c^2}$

$= r_0 |c|$

2) $\sigma(t) = (r_0 \cdot \cos(\frac{t}{r_0}), r_0 \cdot \sin(\frac{t}{r_0}))$

$\sigma'(t) = (-\sin(\frac{t}{r_0}), \cos(\frac{t}{r_0})) \quad \Rightarrow \| \sigma'(t) \| = 1$

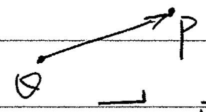
$T(t) = \sigma'(t)$

$k = \| \frac{dT}{dt} \| = \| \sigma''(t) \|$

$\sigma''(t) = (-\frac{1}{r_0} \cos(\frac{t}{r_0}), -\frac{1}{r_0} \sin(\frac{t}{r_0})) \Rightarrow k = \| \frac{dT}{dt} \| = \frac{1}{r_0}$

Midterm · Summary

Vectors



$\vec{v} = P - Q$

$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

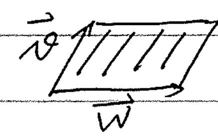
$\vec{v} \cdot \vec{w}$ dot product

$\Rightarrow$ can detect if $\vec{v} \perp \vec{w}$, $\vec{v} \cdot \vec{w} = 0$

$\Rightarrow$ find angle between $\vec{v}$ & $\vec{w}$.

$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\langle \vec{v}, \vec{w} \rangle$.

$\vec{v} \times \vec{w}$ cross product.



$\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \|\vec{w}\| \sin\langle \vec{v}, \vec{w} \rangle$

= area of parallelogram

$(\vec{v} \times \vec{w}) \perp \vec{v}$ & $(\vec{v} \times \vec{w}) \perp \vec{w}$

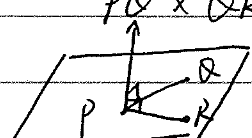
Rt-hand Rule

$\vec{v} \times \vec{w} = \det \begin{vmatrix} i & j & k \\ - & v & - \\ - & w & - \end{vmatrix}$

application * Find equation of plane containing P, Q, R

equation of plane: $n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$.

$\vec{PQ} \times \vec{QR} \Rightarrow$ normal is (n_1, n_2, n_3)

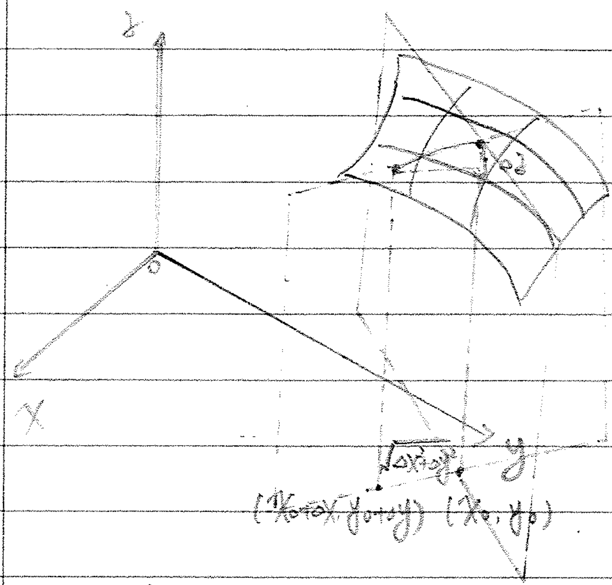


Level sets – graphing surfaces

Polar, cylindrical, spherical coordinates

Curves: velocity, acceleration, curvature

10.10 Chapter 15 §15.1 Introduction to Partial Derivatives



* problem:

Choose different plane

will get different slope.

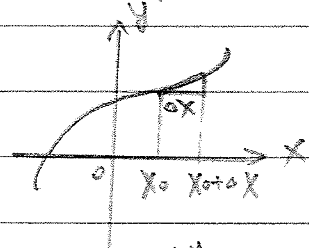
⇓

We need to define

the direction of derivative

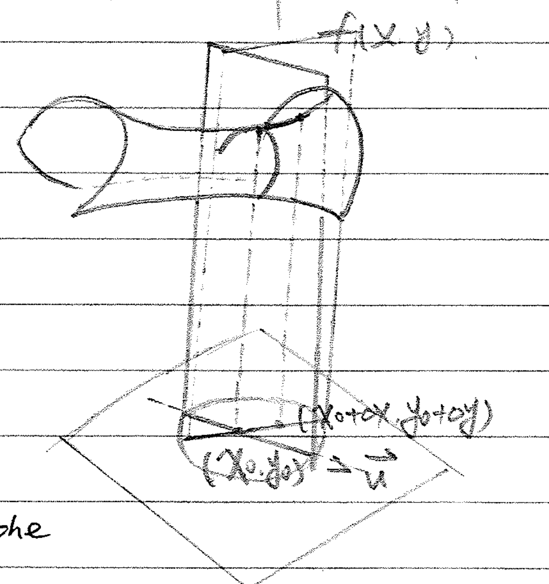
* directional derivative

Function of single variable



$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

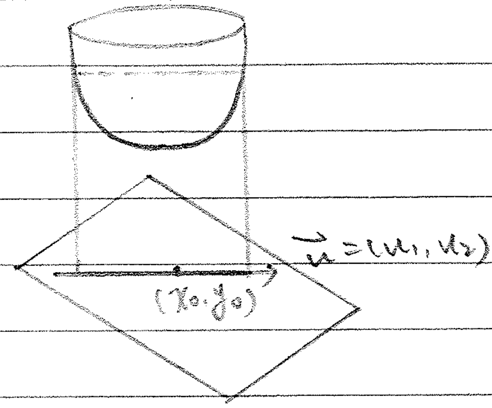
Function of two variables



Directional Derivative:

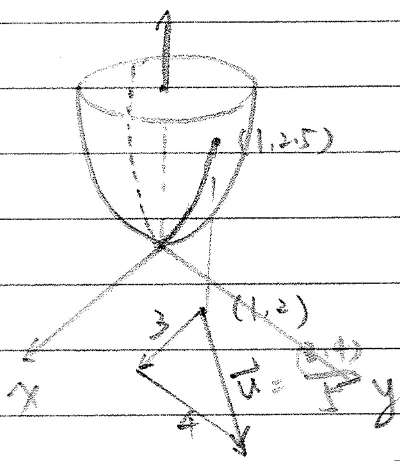
Let $\vec{u}$ be a unit vector in the domain at (x_0, y_0), the directional derivative of $f(x, y)$ at (x_0, y_0) in the direction of $\vec{u}$ is:

$$\left.\frac{df(x_0+tu_1, y_0+tu_2)}{dt}\right|_{t=0}$$



$$\sigma(t) = (x_0, y_0) + t \cdot \vec{u}$$
$$= (x_0, y_0) + t(u_1, u_2)$$
$$= (x_0 + tu_1, y_0 + tu_2)$$
$$\Downarrow$$
$$\left.\frac{df(x_0 + tu_1, y_0 + tu_2)}{dt}\right|_{t=0}$$

Ex. Find the slope of $(1,2,5)$ on the curve $z = x^2 + y^2$.



$$(x(t), y(t)) = (1,2) + t \cdot \left(\tfrac{3}{5}, \tfrac{4}{5}\right)$$
$$= \left(1 + \tfrac{3}{5}t,\ 2 + \tfrac{4}{5}t\right)$$

Substituting $x(t), y(t)$ into

$$z = x^2 + y^2$$

$$\Rightarrow z(t) = \left(1 + \tfrac{3}{5}t\right)^2 + \left(2 + \tfrac{4}{5}t\right)^2$$

$$\Rightarrow z'(t) = 2 \times \left(1 + \tfrac{3}{5}t\right) \cdot \tfrac{3}{5} + 2 \times \left(2 + \tfrac{4}{5}t\right) \cdot \tfrac{4}{5}$$

$$\Rightarrow z'(0) = \frac{22}{5}.$$

* Important: $\vec{u}$ must be a unit vector.

Partial Derivative $\rightarrow$ $\vec{u} = (1,0)$ & $(0,1)$.

D. the partial derivative of $f(x,y)$ of (x_0, y_0) with respect to x is the directional derivative in the direction $\vec{u} = (1,0)$.

$$(x(t), y(t)) = (1,2) + t(1,0)$$
$$= (1+t, 2)$$

Substituting $x(t), y(t)$ into $z = x^2+y^2$

$$z(t) = (1+t)^2 + 4$$
$$z'(t) = 2(1+t)$$
$$z'(0) = 2$$

$\Rightarrow$ the partial derivative of $f(x,y)$ of (x_0, y_0) with respect to y is the directional derivative in the direction $\vec{u} = (0,1)$

$$(x(t), y(t)) = (1,2) + t(0,1)$$
$$= (1, 2+t)$$

Substituting $x(t), y(t)$ into $z = x^2+y^2$

$z(t) = 1+(2+t)^2$ $\qquad$ $z(y) = 1+(2+y)^2$

$z'(t) = 2\times(2+t)$ $\quad\Rightarrow\quad$ $z'(y) = 2\times(2+y)$

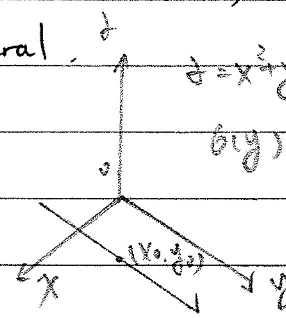
$z'(0) = 4$ $\qquad$ $z'(0) = 4$

$$\Downarrow$$

$$(x, y) = (1, y)$$
$$z = 1+y^2$$
$$z'(y) = 2y$$

$z'(2) = 4 \rightarrow$ the slope of $(1,2) \rightarrow y=2$

General.



$z = x^2+y^2$

$G(y) = (x_0, y)$

$z = x_0^2 + y^2$

$$\frac{dz}{dy} = 2y$$

derivative $= 2y_0$

Partial Derivative

with respect to x: $f_x = \dfrac{\partial z}{\partial x}$

with respect to y: $f_y = \dfrac{\partial z}{\partial y}$

Ex. $z = x^2 + 2xy + y^2$

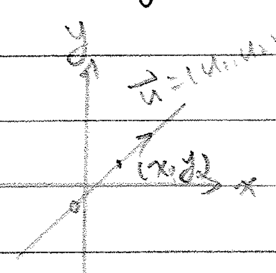
$$\begin{cases} \dfrac{\partial z}{\partial x} = 2x + 2y \\ \dfrac{\partial z}{\partial y} = 2x + 2y \end{cases}$$

* Directional derivative of $f(x, y)$ at (x_0, y_0) in the direction of $\vec{u}$ is: $\dfrac{\partial f}{\partial x} \cdot u_1 + \dfrac{\partial f}{\partial y} \cdot u_2$

Ex. $z = x^2 + y^2$

$\dfrac{\partial z}{\partial x} = \underline{2x}$

$\dfrac{\partial z}{\partial y} = \underline{2y}$



$T = (x_0, y_0) + t(u_1, u_2)$
$= (x_0 + tu_1, y_0 + tu_2)$

$$z(t) = (x_0 + tu_1)^2 + (y_0 + tu_2)^2$$

$$z'(t) = 2(x_0 + tu_1) \cdot u_1 + 2(y_0 + tu_2) \cdot u_2$$

$$z'(0) = \underline{2x_0} \cdot u_1 + \underline{2y_0} \cdot u_2$$

$$= \frac{\partial f}{\partial x} \cdot u_1 + \frac{\partial f}{\partial y} \cdot u_2$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \vec{u}$$