11.2 Chapter 16 §16.3 Maxima and Minima

* Maximum - Minimum Test for Quadratic Functions.

$$f(x,y) = Ax^2 + 2Bxy + Cy^2$$

$$\nabla f = (2Ax + 2By,\ 2Bx + 2Cy)$$

When $\nabla f\big|_{(0,0)} = 0$. Is $(0,0)$ a local minimum or maximum?

$$f = A\left(x^2 + \frac{2B}{A}xy + \frac{B^2y^2}{A^2}\right) + Cy^2 - \frac{B^2}{A}y^2$$

$$= A\left(x + \frac{B}{A}y\right)^2 + y^2\left(\frac{AC-B^2}{A}\right)$$

Suppose $A>0$ $\quad A>0,\ \left(x+\frac{B}{A}y\right)^2 \geq 0,\ y^2 \geq 0,\ \frac{1}{A} > 0.$

if $AC-B^2 > 0$

When $x, y$ changes $A\left(x+\frac{B}{A}y\right)^2 \uparrow$. $y^2\left(\frac{AC-B^2}{A}\right) \uparrow$

$\therefore$ $f(x,y)$ has a local minimum at $(0,0)$

if $AC-B^2 < 0$. $\quad A\left(x+\frac{B}{A}y\right)^2 \uparrow$ $\quad y^2\left(\frac{AC-B^2}{A}\right) \downarrow$

$(0,0)$ is a saddle point

Suppose $A<0$

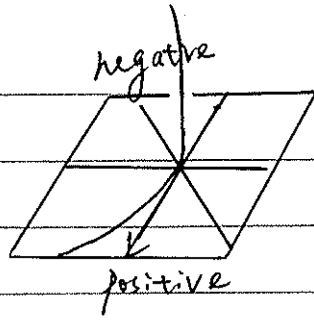
if $AC-B^2 > 0$. $\quad A\left(x+\frac{B}{A}y\right)^2 \downarrow$. $\quad y^2\left(\frac{AC-B^2}{A}\right) \downarrow$

$f(x,y)$ has a local maximum at $(0,0)$

if $AC-B^2 < 0$ $\quad A\left(x+\frac{B}{A}y\right)^2 \downarrow$ $\quad y^2\left(\frac{AC-B^2}{A}\right) \uparrow$

$(0,0)$ is a saddle point

Claim: When $(0,0)$ is neither a minimum nor a maximum point, it's a saddle point.



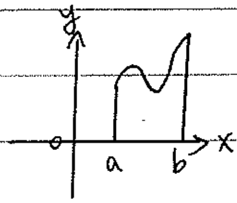

pick a point $(x, y)$

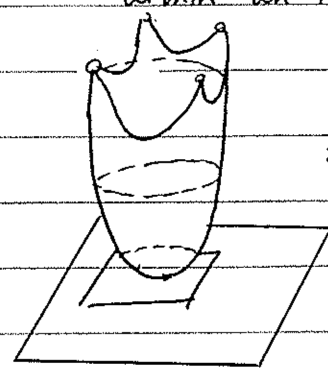
with $x + \frac{By}{A} = 0$

$y = -\frac{A}{B}x$

suppose $A > 0$

* a local maximum $\rightarrow$ a global maximum within an interval / area



$z = x^2 + y^2$

In conclusion

for $f(x, y) = Ax^2 + 2Bxy + Cy^2$

$AC - B^2 > 0$ & $A > 0$ $\Rightarrow$ local minimum

$AC - B^2 > 0$ & $A < 0$ $\Rightarrow$ local maximum

$AC - B^2 < 0$ $\Rightarrow$ saddle point

$AC - B^2 = 0$ $\Rightarrow$ test fails

* Second Derivative Test

$\cancel{f(x, y) = Ax^2 + 2Bxy + Cy^2}$

$f_x = 2Ax + 2By \qquad f_{xx} = 2A \qquad A = \frac{1}{2} f_{xx}$

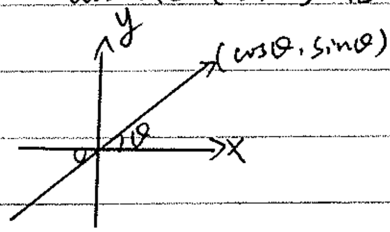
$f_y = 2Bx + 2Cy \qquad f_{yy} = 2C \qquad C = \frac{1}{2} f_{yy}$

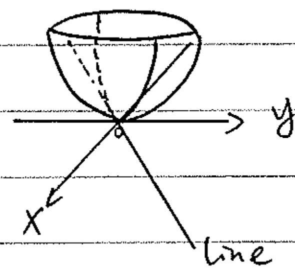
$f_{xy} = 2B \qquad B = \frac{1}{2} f_{xy} = \frac{1}{2} f_{yx}$

$AC - B^2 = \frac{1}{2} f_{xx} \cdot \frac{1}{2} f_{yy} - \left(\frac{1}{2} f_{xy}\right)^2 = \frac{1}{4}\left[f_{xx} f_{yy} - (f_{xy})^2\right]$

Let $f(x,y)$ be any function

assume $(0,0)$ is a critical point & $f(0,0)=0$




let $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

fix $\theta$ & vary $r$

$f(r\cos\theta,\ r\sin\theta)$

$\frac{df}{dr}=0$ at $r=0$ for every $\theta$

$$\frac{df}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dr} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dr} = f_x\cdot\cos\theta + f_y\sin\theta$$

$$\frac{d^2f}{\partial r^2} = \left(f_{xx}\frac{dx}{dr} + f_{xy}\frac{dy}{dr}\right)\cdot\cos\theta + \left(f_{yx}\cdot\frac{dx}{dr} + f_{yy}\cdot\frac{dy}{dr}\right)\cdot\sin\theta$$

$$= f_{xx}\cdot\cos^2\theta + 2f_{xy}\cdot\sin\theta\cdot\cos\theta + f_{yy}\cdot\sin^2\theta$$

$$= \cos^2\theta\cdot f_{xx}(r\cos\theta, r\sin\theta) + 2\sin\theta\cos\theta f_{xy}(r\cos\theta, r\sin\theta)$$
$$+ \sin^2\theta f_{yy}(r\cos\theta, r\sin\theta)$$

think $\theta$ as variable $\quad Ax^2+2Bxy+Cy^2 \quad \begin{array}{l} x\to\cos\theta \\ y\to\sin\theta \end{array}$

$$\Rightarrow f_{xx}\cdot x^2 + 2f_{xy}\cdot xy + f_{yy}\cdot y^2$$

$$\therefore \begin{cases} f_{xx} = A \\ f_{xy} = B \\ f_{yy} = C \end{cases}$$

in conclusion - Second Derivative Test

Let $f(x,y)$ have continuous second partial derivatives, and suppose that $(x_0, y_0)$ is a critical point for $f$:

$$f_x(x_0, y_0) = 0 \quad \& \quad f_y(x_0, y_0) = 0$$

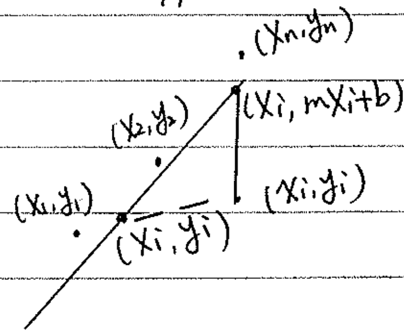
Let $A = f_{xx}(x_0, y_0)$, $B = f_{xy}(x_0, y_0)$, $C = f_{yy}(x_0, y_0)$

| If: | then: |
| --- | --- |
| $A > 0$, $AC - B^2 > 0$ | $(x_0, y_0)$ is a local minimum |
| $A < 0$, $AC - B^2 > 0$ | $(x_0, y_0)$ is a local maximum |
| $AC - B^2 < 0$ | $(x_0, y_0)$ is a saddle point |
| $AC - B^2 = 0$ | the test is inconclusive |

11.5 Chapter 16 §16.3 Maxima and Minima & §16.4

Constrained Extrema and Lagrange Multipliers

§16.3 The method of least squares finds a straight line which "best" approximates a set of data.




$y = mx+b$

has no maxima!

$$0 \le E(m,b) = \sum_{i=1}^{n} (mx_i+b-y_i)^2$$

$$\frac{\partial E}{\partial m} = \sum_{i=1}^{n} 2(mx_i+b-y_i)\cdot x_i$$

$$\frac{\partial E}{\partial b} = \sum_{i=1}^{n} 2(mx_i+b-y_i)$$

$\frac{\partial E}{\partial m} = 0$ & $\frac{\partial E}{\partial b} = 0$ give

$$\frac{\partial E}{\partial m} = 2\left(m\sum x_i^2 + b\sum x_i - \sum x_i y_i\right); \quad \frac{\partial E}{\partial b} = 2\left(m\sum x_i + nb - \sum y_i\right)$$

$$\Rightarrow \left.\begin{array}{ll} m\sum x_i^2 + b\sum x_i - \sum x_i y_i = 0 & ① \\ m\sum x_i + n\,b - \sum y_i = 0 & ② \end{array}\right\}$$

divide both equations by $n$

$$\Rightarrow \begin{cases} m\cdot\frac{\sum x_i^2}{n} + b\,\frac{\sum x_i}{n} - \frac{\sum x_i y_i}{n} = 0 \\ m\cdot\frac{\sum x_i}{n} + b - \frac{\sum y_i}{n} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} m\cdot\overline{x_i^2} + b\overline{x_i} - \overline{x_i y_i} = 0 & ① \\ m\overline{x_i} + b - \overline{y_i} = 0 \Rightarrow (\overline{x_i}, \overline{y_i}) \text{ is on } y = mx+b. \end{cases}$$

$\Rightarrow b = \overline{y_i} - m\overline{x_i}$ substituting into ①

$$m\overline{x_i^2} + (\overline{y_i} - m\overline{x_i})\cdot\overline{x_i} - \overline{x_iy_i} = 0$$

$$\Rightarrow m = \frac{\overline{x_iy_i} - \overline{x_i}\cdot\overline{y_i}}{\overline{x_i^2} - (\overline{x_i})^2}$$

Ex.

| $x$ | $y$ | $x^2$ | $xy$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 3 | 4 | 6 |
| 3 | 6 | 9 | 18 |
| 5 | 7 | 25 | 35 |

$\overline{x_i} = \frac{11}{4}$, $\overline{y_i} = \frac{17}{4}$, $\overline{x_i^2} = \frac{39}{4}$, $\overline{x_iy_i} = 15$

$$m = \frac{15 - \frac{11}{4}\times\frac{17}{4}}{\frac{39}{4} - \frac{121}{16}} = \frac{\frac{53}{16}}{\frac{35}{16}} = \frac{53}{16}\times\frac{16}{35} = \frac{53}{35}.$$

$$b = \frac{17}{4} - \frac{53}{35}\cdot\frac{11}{4} = \frac{12}{140} = \frac{3}{35}.$$

$$\therefore f = \frac{53}{35}x + \frac{3}{35}.$$

* Supplement.

$05^2 = \overset{0\times1}{}|25$

$15^2 = \overset{1\times2}{2}|25$

$25^2 = \overset{2\times3}{6}|25$

$35^2 = \overset{3\times4}{12}|25$

$45^2 = \overset{4\times5}{20}|25$

$55^2 = \overset{5\times6}{30}|25$

$65^2 = \overset{6\times7}{42}|25$

$75^2 = \overset{7\times8}{56}|25$

$85^2 = \overset{8\times9}{72}|25$

$95^2 = \overset{9\times10}{90}|25$

$$(10n+5)^2 = 100n^2 + 100n + 25$$
$$= 100n(n+1) + 25$$

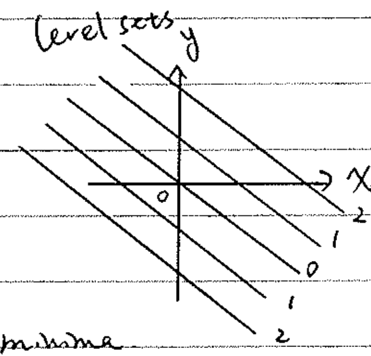
* 2nd derivative test

$f(x,y)$ $(x_0,y_0)$ is a critical point

$$AC-B^2 = f_{xx}(x_0,y_0)\cdot f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2 > 0 \quad \begin{cases} \& \ f_{xx}(x_0,y_0) > 0 & \text{Min} \\ \& \ f_{xx}(x_0,y_0) < 0 & \text{Max} \end{cases}$$

$AC-B^2 < 0$ saddle

$AC-B^2 = 0$ test fails why

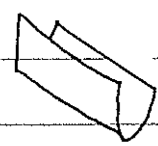



Ex. $f(x,y) = x^2+2xy+y^2 = (x+y)^2 \geq 0$

$\nabla f = (2x+2y,\ 2x+2y) = (0,0)$

$y=-x \Rightarrow$ minima

Graph of $f$



$A = f_{xx} = 2$

$B = f_{xy} = 2$ $\quad AC-B^2=0$.

$C = f_{yy} = 2$

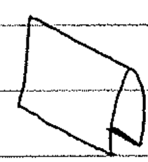
However, $g(x,y) = -f(x,y)$

$A=-2$

$B=-2$

$C=-2$

$AC-B^2=0$



all critical points are maxima.

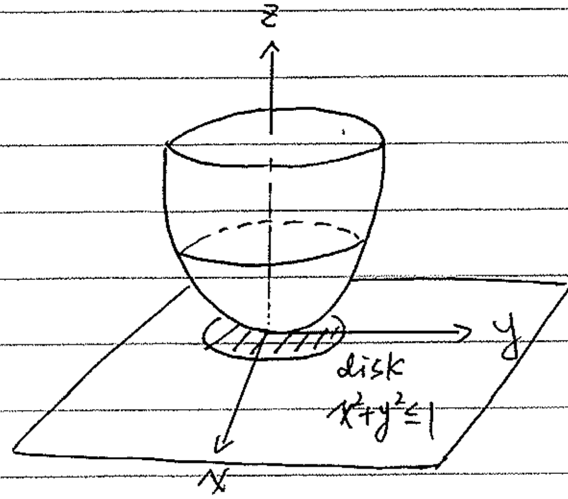
## §16.4 Constrained Extrema & Lagrange Multipliers

Problem: minimize or maximize some function

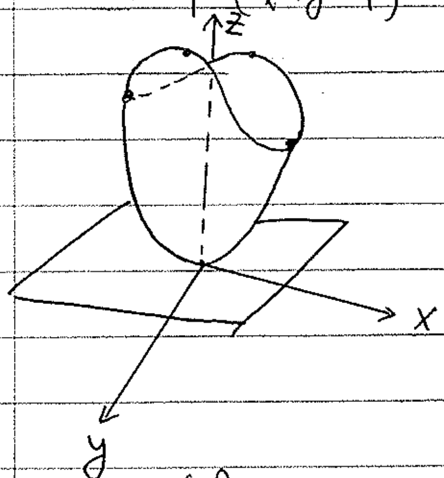
$f(x,y)$ subject to a constrain $g(x,y)=c$

Ex. $f(x,y) = x^2 + 2y^2 = a$

$$\frac{x^2}{(\sqrt{a})^2} + \frac{y^2}{\left(\frac{\sqrt{a}}{2}\right)^2} = 1$$

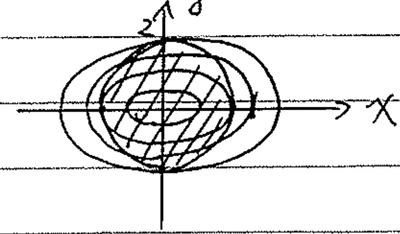



Graph$(x^2+y^2 \le 1)$




level sets

domain $(x^2+y^2 \le 1)$




Way 1: parametrize the boundary of the domain

domain: $x^2+y^2=1$ $\quad \sigma(t) = (\cos t, \sin t)$

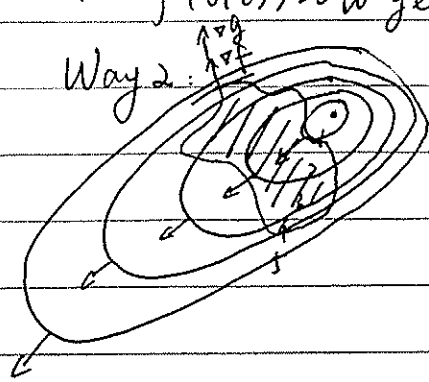
$f(\sigma(t)) = \cos^2 t + 2\sin^2 t = 1 + \sin^2 t$

$f'(\sigma(t)) = 0 \Rightarrow$ get possible maxima & minima points

evaluating $f$ at the critical points and where

$f'(\sigma(t)) = 0$ to get the maxima & minima of $f$.

Way 2:




$g(x,y) = c \rightarrow$ look for critical point inside.

$$\Rightarrow \begin{cases} g(x,y) = c \\ \nabla g = \lambda \nabla f \end{cases}$$

Method of Lagrange Multipliers

11.7 HW §16.4 (13) (9)

§16.4 Constrained Extrema & Lagrange Multipliers

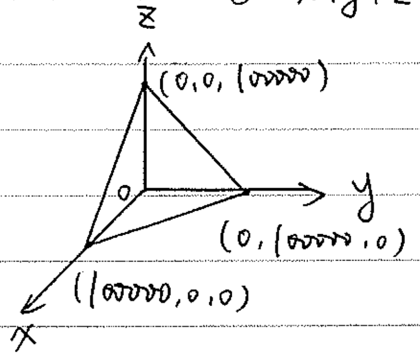
To find the extreme points of $f(x,y)$ subject to the constraint $g(x,y)=C$

13. $R(x,y,z) = 8xyz^2 - 200000(x+y+z)$

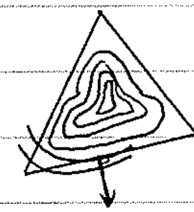
$x+y+z = 100000$

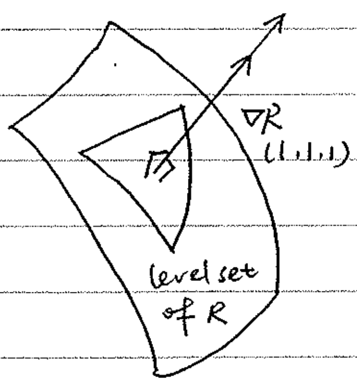
How to maximize $R$?

domain is $x \geq 0,\ y \geq 0,\ z \geq 0$.
& $x+y+z = 100000$




$\Rightarrow$ We try to find the maxima value at the point when the gradient of the level set // the vector perpendicular to the edge.






$$\begin{cases} R(x,y,z) = 8xyz^2 - 200000(x+y+z) \\ g(x,y,z) = x+y+z \end{cases}$$

$\Downarrow$

$$\begin{cases} \nabla R = \lambda \nabla g \\ g(x,y,z) = 100000. \end{cases}$$

$$\nabla R = (8yz^2 - 200000,\ 8xz^2 - 200000,\ 16xyz - 200000) = \lambda(1,1,1)$$

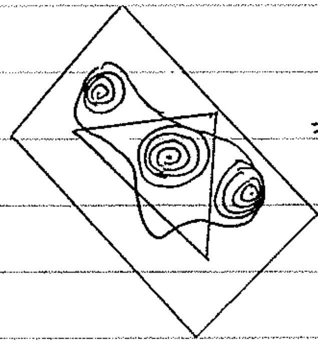
$$\Rightarrow \begin{cases} x+y+z = 100000 & ① \\ 8yz^2 - 200000 = \lambda & ② \\ 8xz^2 - 200000 = \lambda & ③ \\ 16xyz - 200000 = \lambda & ④ \\ x \geq 0,\ y \geq 0,\ z \geq 0 & ⑤ \end{cases}$$

From ② & ③ $\Rightarrow$ $y = x$ ⑥

Substituting ⑥ into ④, From ④ & ③

$\Rightarrow 8xz^2 = 16x^2z \Rightarrow z = 2x$

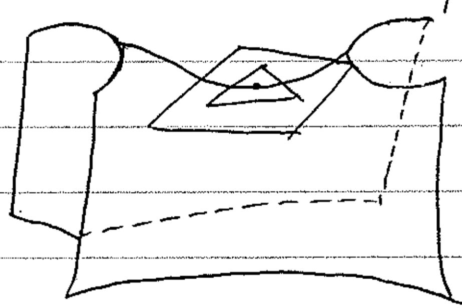
$\therefore x = y = \frac{1}{2}z \Rightarrow \begin{cases} x = 25000 \\ y = 25000 \\ z = 50000 \end{cases}$

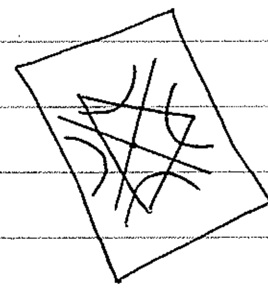


$$R(25000, 25000, 50000)$$
$$= 8 \times 25000 \times 25000 \times 50000^2 - 2\times10^5\times10^5$$
$$= 200 \times 25^2 \times 10^{14} - 2\times10^{10}$$
$$= 1.25\times10^{19} - 2\times10^{10}$$

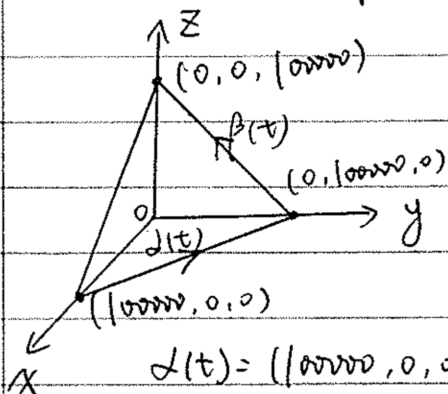
Check: if $R_{(25000,\,25000,\,50000)}$ is the maxima point of $R(x,y,z)$?

* $R$ could be a saddle point.





parametrize the boundary to check if $R_{(25000,\,25000,\,50000)}$ is the maxima point of $R(x,y,z)$ if $x+y+z = 100000$.




$$\beta(t) = (0, 10^5, 0) + t(0, -10^5, 10^5)$$
$$= (0, 10^5(1-t), 10^5 t)$$

$$\alpha(t) = (100000, 0, 0) + t(-100000, 100000, 0)$$
$$= (100000 - t\cdot100000,\ 100000t,\ 0)$$

$$\cancel{R(\alpha(t)) = 8\cdot10^5(1-t)\,10^5 t\cdot 0 - 2\times10^{10}}$$

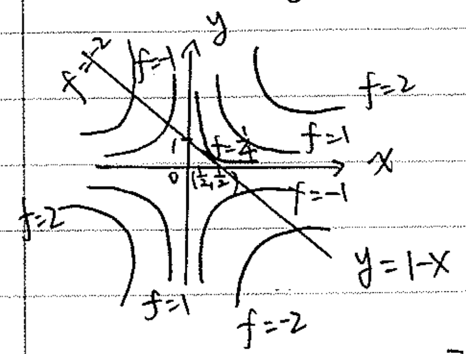
$$\beta(\alpha(t)) = 0 - 2\times10^{10}$$

...

9. Find the extrema of $f$ subject to the stated constrains in Ex 9.

$f(x,y) = xy$; $x+y=1$

* ($\lambda$ cannot equal to 0)




$xy = c$, $y = \frac{c}{x}$ → level set.

$\Rightarrow$ domain $y = 1-x$

< no minima > → observe through the level-set graph of $f(x,y)$)

$\Rightarrow \begin{cases} \nabla f = \lambda \cdot \nabla g \\ x+y=1 \end{cases}$ $\begin{cases} f(x,y) = xy \\ g(x,y) = x+y = 1 \end{cases}$

$\nabla f = (y,x)$ $\Rightarrow \begin{cases} (y,x) = \lambda(1,1) \\ x+y=1 \end{cases} \Rightarrow \begin{cases} y = \lambda \\ x = \lambda \\ x+y=1 \end{cases} \Rightarrow x = y = \lambda = \frac{1}{2}$.