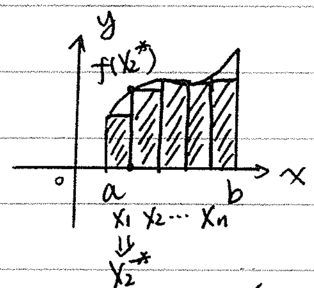
11-12. Chapter 17. Multiple Integration

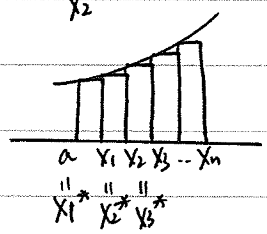
§17.1 The Double Integral & Iterated Integral.

→ one variable – Riemann Sums < $y=f(x)$ >

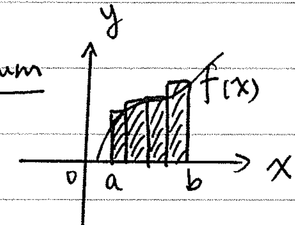


in an interval $[a,b]$

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) = \sum f(x_i^*)\,\Delta x_i$$

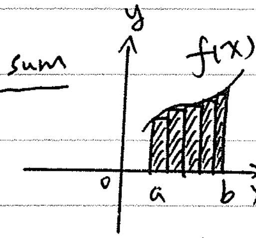


upper sum



the sum > the area below $f(x)$ within interval $[a,b]$

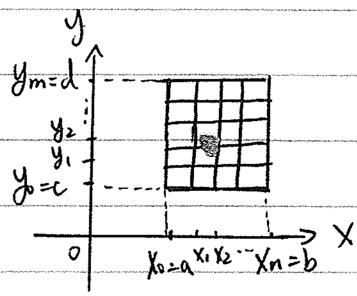
lower sum

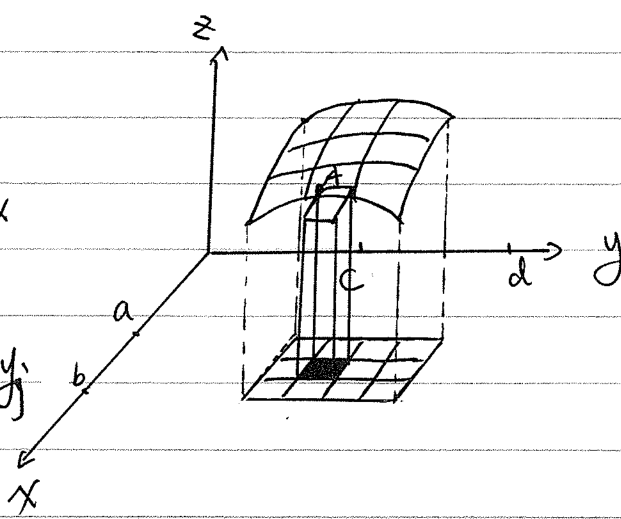


the sum < the area below $f(x)$ within interval $[a,b]$

→ two variables < $z=f(x,y)$ >

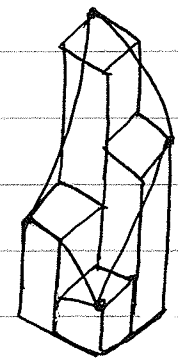
one variable	v.s.	two variables
$y=f(x)$		$z=f(x,y)$
interval $[a,b]$		closed rectangle $D=[a,b]\times[c,d]$





$$\sum f(x_i^*, y_j^*)\,\Delta x_i\,\Delta y_j$$

upper & lower sum



lower sum.

$$\sum f(x_i^*, y_j^*)\, \Delta x_i \cdot \Delta y_j$$

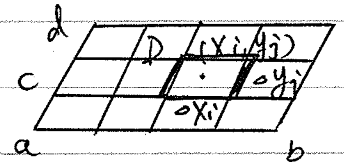
$\downarrow$ limit

$$\iint_D f(x,y)\,dx\,dy$$

double integral.

$1 \le i \le n$

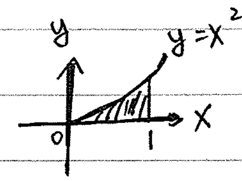
$1 \le j \le m$



$D = [a,b] \times [c,d]$ in the domain of $f(x,y)$

Ex.

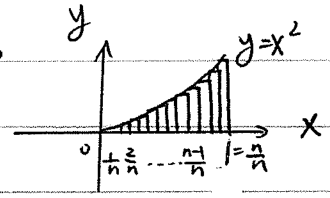
① . $f(x) = x^2$.



Calculate the area under $y = x^2$ within the interval $[0,1]$

<1> $\int_0^1 x^2\,dx = \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{3}$.

<2>



$$S = 0^2 \cdot \frac{1}{n} + \left(\frac{1}{n}\right)^2 \cdot \frac{1}{n} + \left(\frac{2}{n}\right)^2 \cdot \frac{1}{n} + \cdots + \left(\frac{n-1}{n}\right)^2 \cdot \frac{1}{n}$$

$$= \frac{1^2}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{(n-1)^2}{n^3}$$

$$= \frac{1}{n^3}\left[1^2 + 2^2 + \cdots + (n-1)^2\right]$$

$$\boxed{* \text{ formula } \quad 1^2 + 2^2 + \cdots + m^2 = \frac{m(m+1)(2m+1)}{6}}$$

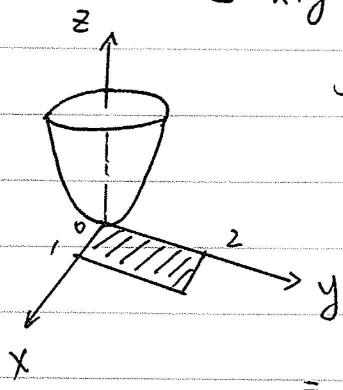
$$= \frac{1}{n^3} \cdot \frac{(n-1)n \cdot (2n-2+1)}{6}$$

$$= \frac{(n-1)(2n-1)}{6n^2}$$

$$S = \lim_{n \to \infty} \frac{(n-1)\cdot(2n-1)}{6n^2} = \frac{2n^2 - 3n + 1}{6n^2}$$

$$= \frac{2 - \frac{3}{n} + \frac{1}{n^2}}{6} = \frac{1}{3}$$

EX ②. $z = x^2+y^2$



$\iint_D x^2+y^2 \, dx\,dy \quad D = [0,1] \times [0,2]$

<1>. $\iint_D x^2+y^2 \, dx\,dy$ treat y as constant

$= \int_0^2 \left[\int_0^1 (x^2+y^2) \cdot dx \right] \cdot dy$

$= \int_0^2 \left[\frac{1}{3}x^3 + y^2 x \Big|_0^1 \right] \cdot dy$

$= \int_0^2 \left(\frac{1}{3} + y^2 \right) \cdot dy$

$= \frac{1}{3}y + \frac{1}{3}y^3 \Big|_0^2$

$= \frac{2}{3} + \frac{8}{3}$

$= \frac{10}{3}.$

<2> $\int_0^1 \left[\int_0^2 (x^2+y^2) \cdot dy \right] \cdot dx$

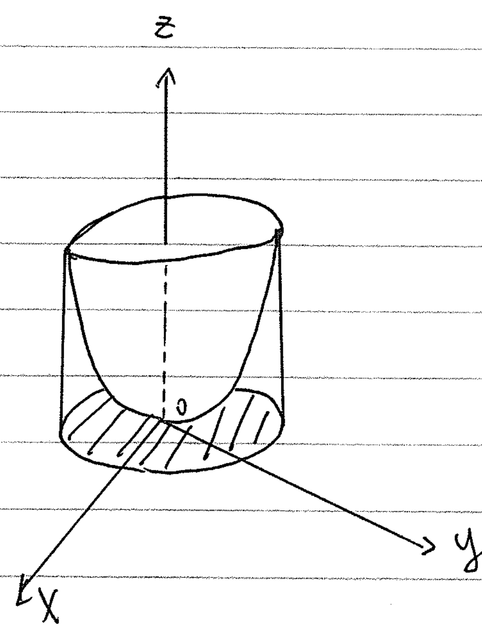
$= \int_0^1 \left[x^2 y + \frac{1}{3}y^3 \Big|_0^2 \right] \cdot dx$

$= \int_0^1 \left(2x^2 + \frac{8}{3} \right) \cdot dx$

$= \frac{2}{3}x^3 + \frac{8}{3}x \Big|_0^1$

$= \frac{2}{3} + \frac{8}{3}$

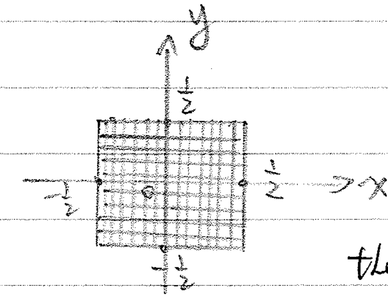
$= \frac{10}{3}.$



11.14 Chapter 17 Multiple Integration

§17.2 The Double Integral Over General Regions

HW §17.1 21)



The density at each point of a 1 centimeter square microchip is $4+r^2$ grams per square centimeter, where r is the distance in centimeters from the point to the center of the chip. What is the mass of the chip?

$$\begin{cases} r^2 = x^2+y^2 \\ \rho = 4+r^2 = 4+x^2+y^2 \end{cases}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} 4+x^2+y^2 \cdot dx\, dy$$

$$= 4\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} 4+x^2+y^2 \cdot dx\, dy$$

$$= 4\int_0^{\frac{1}{2}} 4x+\frac{1}{3}x^3+xy^2 \Big|_0^{\frac{1}{2}} \cdot dy$$

$$= 4\int_0^{\frac{1}{2}} \left(2+\frac{1}{24}+\frac{1}{2}y^2\right) \cdot dy$$

$$= 4\left(\frac{49}{24}y+\frac{1}{6}y^3 \Big|_0^{\frac{1}{2}}\right)$$

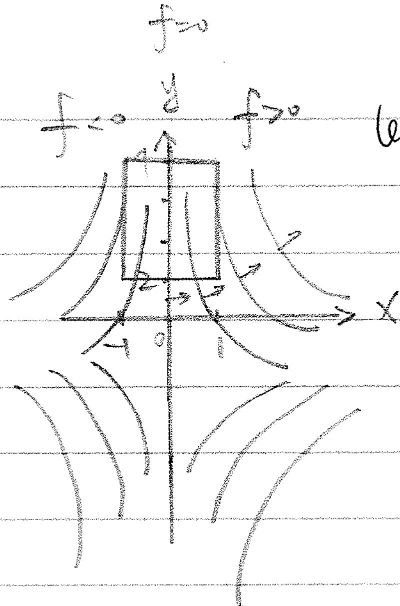
$$= 4 \times \left(\frac{49}{48}+\frac{1}{48}\right)$$

$$= \frac{25}{6}$$

3). $-1 \le x \le 1$ & $2 \le y \le 4$. $f(x,y) = x(1+y)$

Find step function $g(x,y) \le f(x,y)$ to show that

$$\iint_D f(x,y)\, dx\, dy \ge -9$$



level set.

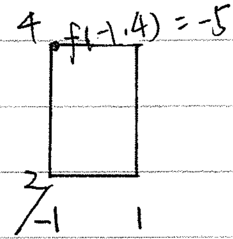
$$f(x,y) = x(1+y).$$

$$x(1+y) = c$$

$$1+y = \frac{c}{x}$$

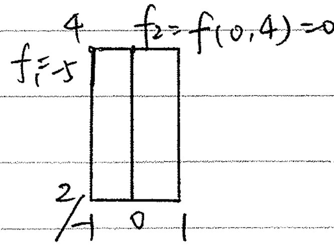
$$y = \frac{c}{x} - 1.$$

1>



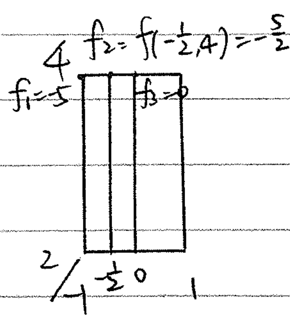
$$\iint_D g(x,y)\,dx\,dy = -20 < -9 \quad \times$$

2>



$$\iint_D g(x,y)\,dx\,dy = -5\times 2 + 0\times 2 = -10 < -9 \quad \times$$

3>



$$\iint_D g(x,y)\,dx\,dy = -5\times 1 - \frac{5}{2}\times 1 + 0\times 2 = -7.5 > -9 \quad \checkmark$$

$$\because\ g(x,y) \le f(x,y)$$

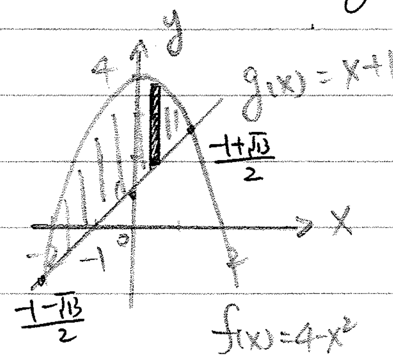
$$\therefore\ f(x,y) \ge -9.$$

$$\Rightarrow\ -9 \le \iint_D g(x,y)\,dx\,dy \le \iint_D f(x,y)\,dx\,dy$$

§17.2 The Double Integral Over General Regions

Ex. $f(x) = 4 - x^2$

$g(x) = x + 1$



$\delta(x,y) = y^2$

→ use small rectangle to analyze the regions

when

$4 - x^2 = x + 1$

$x^2 + x - 3 = 0$

$\Delta = 1 - 4 \times 1 \times (-3) = 13$

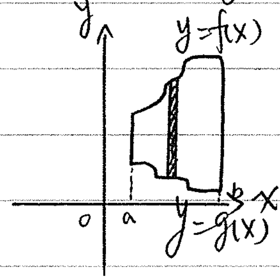
$x_1, x_2 = \frac{-1 \pm \sqrt{13}}{2}$

⇒ The Region.

$$\int_{\frac{-1-\sqrt{13}}{2}}^{\frac{-1+\sqrt{13}}{2}} \int_{x+1}^{4-x^2} y^2 \, dy \, dx$$

* Type 1 & Type 2 Region

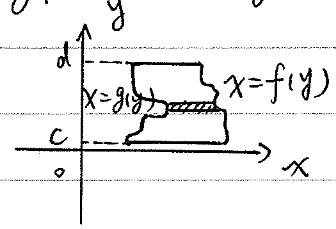
Type 1. Integrate with respect to y first.



→ vertical first

$$\int_a^b \int_{g(x)}^{f(x)} \varphi(x,y) \, dy \, dx.$$

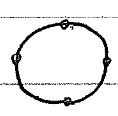
Type 2. Integrate with respect to x first



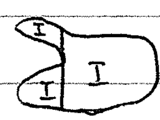
→ horizontal first

$$\int_c^d \int_{g(y)}^{f(y)} \varphi(x,y) \, dx \, dy$$

Some regions are both type 1 & 2, some are neither.



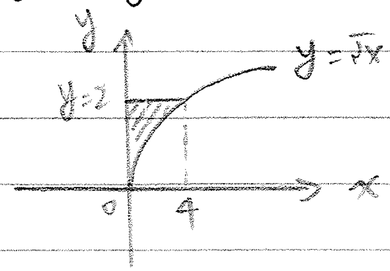
both type 1 & 2



→ we can divide a whole region into several parts

Ex. §17.2 ⑦.

Find $\iint_D (x-y)^2\,dx\,dy$. where D is the region in Figure 17.2.13.



$y = \sqrt{x}$

$/ x = y^2$

Type 1

$$\int_0^4 \int_{x^{\frac{1}{2}}}^2 x^2 - 2xy + y^2 \, dy\, dx$$

$$= \int_0^4 \left(x^2 y - xy^2 + \tfrac{1}{3} y^3 \Big|_{x^{\frac{1}{2}}}^{2} \right) . dx$$

$$= \int_0^4 \left(2x^2 - 4x + \tfrac{8}{3} - x^{\frac{5}{2}} + x^2 - \tfrac{1}{3} x^{\frac{3}{2}} \right) . dx$$

$$= \int_0^4 \left(\tfrac{8}{3} - 4x - \tfrac{1}{3} x^{\frac{3}{2}} + 3x^2 - x^{\frac{5}{2}} \right) dx$$

$$= \frac{8}{3}x - 2x^2 - \frac{2}{15} x^{\frac{5}{2}} + x^3 - \frac{2}{7} x^{\frac{7}{2}} \Big|_0^4$$

$$= \frac{32}{3} - 32 - \frac{2}{15} \times 32 + 64 - \frac{2}{7} \times 128$$

$$= \frac{1120 + 3360 - 448 - 3840}{105} = \frac{192}{105} = \underline{\frac{64}{35}}$$

Type 2.

$$\int_0^2 \int_0^{y^2} x^2 - 2xy + y^2 . \, dx\, dy$$

$$= \int_0^2 \left(\tfrac{1}{3} x^3 - x^2 y + xy^2 \Big|_0^{y^2} \right) . dy$$

$$= \int_0^2 \tfrac{1}{3} y^6 - y^5 + y^4 \, dy$$

$$= \frac{1}{21} y^7 - \frac{1}{6} y^6 + \frac{1}{5} y^5 \Big|_0^2$$

$$= \frac{128}{21} - \frac{64}{6} + \frac{32}{5}$$

$$= \frac{640 - 1120 + 672}{105} = \frac{192}{105} = \underline{\frac{64}{35}}$$

* By using different types to analyze type region we can get same result, but the difficulty of calculation is quite different.

11.16 Midterm & §17.2 HW

Midterm 4

Let $f(x,y) = x^3 - x + y^2$

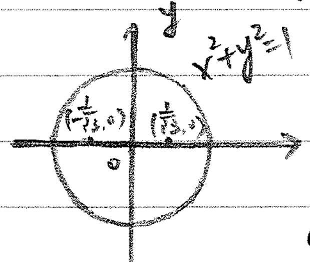
a) Find all critical points of $f(x,y)$ and classify them as local minima, local maxima, or saddle points.

$\nabla f(x,y) = (3x^2-1,\ 2y)$ $\quad \begin{cases} 3x^2-1=0 \\ 2y=0 \end{cases}$ critical points $(\frac{1}{\sqrt{3}}, 0)$ & $(-\frac{1}{\sqrt{3}}, 0)$

$A = f_{xx} = 6x,\quad B = f_{xy} = 0,\quad C = f_{yy} = 2$

Point	A	B	C	$AC-B^2$	classify
$(\frac{1}{\sqrt{3}}, 0)$	$\frac{6}{\sqrt{3}} > 0$	0	2	$\frac{12}{\sqrt{3}} > 0$	minima
$(-\frac{1}{\sqrt{3}}, 0)$	$-\frac{6}{\sqrt{3}} < 0$	0	2	$-\frac{12}{\sqrt{3}} < 0$	saddle

b) Find the maximum & minimum values of $f(x,y)$ on the unit disk $x^2+y^2 \le 1$.



$$f(\tfrac{1}{\sqrt{3}}, 0) = (\tfrac{1}{\sqrt{3}})^3 - \tfrac{1}{\sqrt{3}} + 0$$
$$= -\frac{2}{3\sqrt{3}}$$

$g(x,y) = x^2+y^2 = 1$

$$\begin{cases} g(x,y) = 1 \\ \nabla f = \lambda \nabla g \end{cases} \Rightarrow \begin{cases} x^2+y^2 = 1 \\ (3x^2-1, 2y) = \lambda(2x, 2y) \end{cases}$$

$$\begin{cases} x^2+y^2=1 & ① \\ 3x^2-1=2\lambda x & ② \\ 2y=2\lambda y & ③ \end{cases}$$

from ③ if $y=0$. $\lambda \in R$.

$x=\pm 1$.

if $y\neq 0$ $\lambda=1$

$3x^2-1-2x=0$

$(3x+1)(x-1)=0$, $x=1$ or $x=-\frac{1}{3}$

when $x=1$, $y=0$

$x=-\frac{1}{3}$, $y=\pm\sqrt{1-\frac{1}{9}}=\pm\frac{2}{3}\sqrt{2}$

possible points $(\frac{1}{\sqrt{3}},0)$, $(1,0)$, $(-1,0)$ $(-\frac{1}{3},\frac{2}{3}\sqrt{2})$ & $(-\frac{1}{3},-\frac{2}{3}\sqrt{2})$

$f(\frac{1}{\sqrt{3}},0)=-\frac{2}{3\sqrt{3}}$

$f(1,0)=1^3-1+0=0$

$f(-1,0)=(-1)^3-(-1)+0=0$

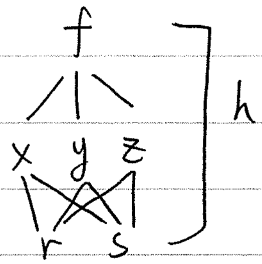
$f(-\frac{1}{3},\frac{2}{3}\sqrt{2})=f(-\frac{1}{3},-\frac{2}{3}\sqrt{2})=-\frac{1}{27}+\frac{1}{3}+\frac{8}{9}=\frac{-1+9+24}{27}=\frac{32}{27}$

$\therefore f(-\frac{1}{3},\frac{2}{3}\sqrt{2})=f(-\frac{1}{3},-\frac{2}{3}\sqrt{2})>f(1,0)=f(-1,0)>f(\frac{1}{\sqrt{3}},0)$

the maximum value is $\frac{32}{27}$, located at points $(-\frac{1}{3},\frac{2}{3}\sqrt{2})$ & $(-\frac{1}{3},-\frac{2}{3}\sqrt{2})$; the minimum value is $-\frac{2}{3\sqrt{3}}$, located at the point $(\frac{1}{\sqrt{3}},0)$.

3. (c) Let $f(x,y,z)=z+x\cdot\sin y$

Let $x=r+s$, $y=r-s$, $z=rs$. In which direction should one travel from the point $(-1,2)$ in the rs-plane so that $h(r,s)=f(r+s,r-s,rs)$ will decrease most rapidly?



$f(x,y,z) = z + x \cdot \sin y$

$x = r+s, \quad y = r-s, \quad z = rs$

Method 1: $h(r,s) = f(r+s,\ r-s,\ rs)$

$$= rs + (r+s)\sin(r-s)$$

$$\nabla h(r,s) = \left(s + \sin(r-s) + (r+s)\cdot\cos(r-s),\ r + \sin(r-s) - (r+s)\cdot\cos(r-s)\right)$$

When the $\vec{d}$ is opposite to $\nabla h(r,s)$, $h(r,s)$ will decrease most rapidly

So:

$$\vec{d} = -\nabla h(-1,2) = -\left(2 + \sin(-3) + \cos(-3),\ -1 + \sin(-3) - \cos(-3)\right)$$

$$= \left(-2 - \sin(-3) - \cos(-3),\ 1 - \sin(-3) + \cos(-3)\right).$$

Method 2:

$$\nabla h = \left(\frac{\partial h}{\partial r}, \frac{\partial h}{\partial s}\right) = \underbrace{\frac{\partial(f)}{\partial(x,y,z)}}_{1\times 3} \cdot \underbrace{\frac{\partial(x,y,z)}{\partial(r,s)}}_{3\times 2}$$

$$= \left(\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r},\ \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial s}\right)$$

$$= (\sin y,\ x\cos y,\ 1)\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ s & r \end{pmatrix}$$

$$\nabla h(-1,2) = (\sin(-3),\ \cos(-3),\ 1)\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & -1 \end{pmatrix}$$

$$= \left(\sin(-3) + \cos(-3) + 2,\ \sin(-3) - \cos(-3) - 1\right)$$

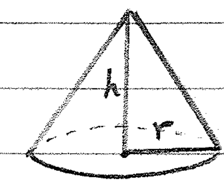
$$\vec{d} = -\nabla h(-1,2) = \left(-2 - \sin(-3) - \cos(-3),\ 1 - \sin(-3) + \cos(-3)\right)$$

2. Cone with height h & base a disk of radius r.

t_1 $h_1 = 10$ feet, $d_1 = 16$ feet

$\frac{dh_1}{dt_1} = 4$ inches/min $\quad \frac{dd_1}{dt_1} = 6$ inches/min

At what rate is the volume growing at that moment?



$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial v}{\partial h}\cdot\frac{dh}{dt}$$
$$= \frac{2}{3}\pi r h\cdot\frac{dr}{dt} + \frac{1}{3}\pi r^2\cdot\frac{dh}{dt}$$

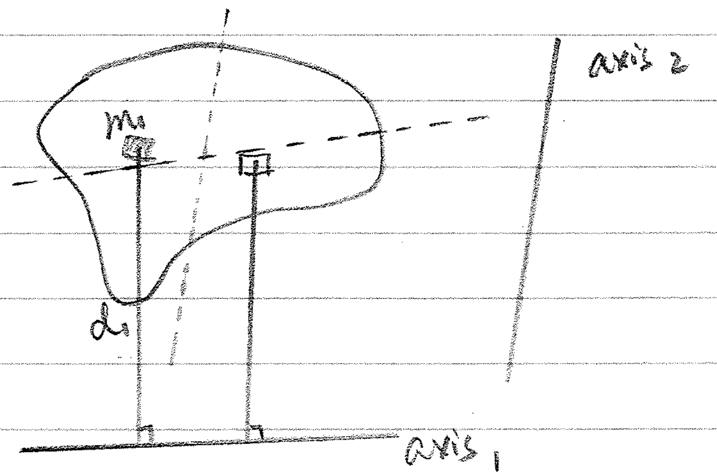
HW §17.2

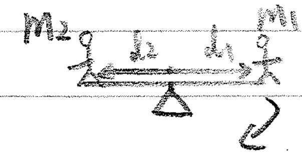
11. $$\int_{-1}^{1}\int_{y^{2/3}}^{(2-y)^2} (y\sqrt{x} + y^3 - 2y)\,dx\,dy$$
$$= \int_{-1}^{1} \frac{2}{3}yx^{\frac{3}{2}} + y^3x - 2yx \Big|_{y^{2/3}}^{(2-y)^2} dy$$
$$= \int_{-1}^{1} \left[\frac{2}{3}y(2-y)^3 + y^3(2-y)^2 - 2y(2-y)^2\right] - \left[\frac{2}{3}y^2 + y^{\frac{11}{3}} - 2y^{\frac{5}{3}}\right] dy$$

9. $$\int_0^{\pi}\int_{\sin x}^{3\sin x} x(1+y)\,dy\,dx$$
$$= \int_0^{\pi} xy + \frac{1}{2}xy^2 \Big|_{\sin x}^{3\sin x} dx$$ Using integration by parts
$$= \int_0^{\pi} 2\sin x\cdot x + 4\sin^2 x\cdot x \rightarrow \int u\,dv = uv - \int v\,du$$

11.19 Chapter 17 Multiple Integration

§17.3 Applications of the Double Integral



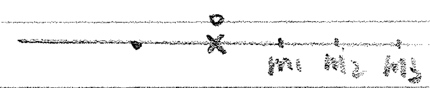


$m_1 d_1 = m_2 d_2$

$0 = m_1 d_1 - m_2 d_2$

$0 = m_1 d_1 + m_2(-d_2)$

The moment = mass × distance.

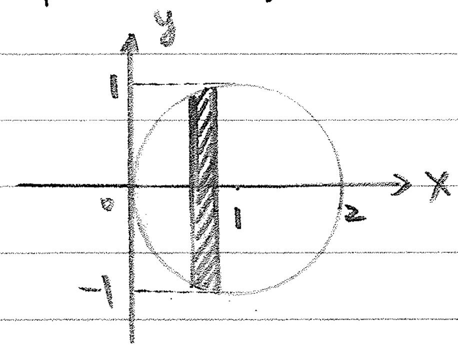


by using two axes, we can find the center of mass, which is:

$$\bar{x} = \frac{\iint_D x\,\rho(x,y)\,dx\,dy}{\iint_D \rho(x,y)\,dx\,dy} \quad \& \quad \bar{y} = \frac{\iint_D y\,\rho(x,y)\,dx\,dy}{\iint_D \rho(x,y)\,dx\,dy} \quad \left(\frac{\text{moment}}{\text{mass}}\right)$$

Ex. HW 21.

Find the center of mass of the disk determined by $(x-1)^2+y^2 \le 1$ if the density is x^2.



$\rho = x^2$

1) $(x-1)^2+y^2 \le 1$

$y^2 = 1-(x-1)^2$

$y = \pm\sqrt{1-(x-1)^2}$

notice that

$\rho(x) = \rho(-x)$

$\Uparrow$

$$\text{mass} = \int_0^2 \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} x^2\,dy\,dx = 2\int_0^2 \int_0^{\sqrt{1-(x-1)^2}} x^2\,dy.$$

$$\text{mass} = 2\int_0^2 x^2 y \Big|_0^{\sqrt{1-(x-1)^2}} dx$$

$$= 2\int_0^2 x^2 \sqrt{1-(x-1)^2}\, dx$$

Assume $x-1 = \sin\theta$ Then $dx = \cos\theta\, d\theta$ $\quad x \in [0,2] \to x-1 \in [-1,1]$ $\to \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

s.t.

$$\text{mass} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\sin\theta)^2 \sqrt{1-\sin^2\theta}\cos\theta\, d\theta$$

when $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ $\quad \sqrt{1-\sin^2\theta} = \cos\theta$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\sin\theta)^2 \cos^2\theta\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+2\sin\theta+\sin^2\theta)\cdot\cos^2\theta\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta + 2\sin\theta\cos^2\theta + \sin^2\theta\cos^2\theta\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2} + 2\sin\theta\cos^2\theta + (1-\cos^2\theta)\cos^2\theta\, d\theta$$

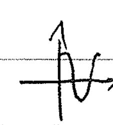
$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1+\cos 2\theta + 2\sin\theta\cos^2\theta - \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1+\cos 2\theta + 2\sin\theta\cos^2\theta - \frac{1}{4} - \frac{1}{4}\cos^2 2\theta - \frac{1}{2}\cos 2\theta\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{3}{4} + \frac{1}{2}\cos 2\theta + 2\sin\theta\cos^2\theta - \frac{1}{4}\times\frac{1+\cos 4\theta}{2}\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{5}{8} + \frac{1}{2}\cos 2\theta + 2\sin\theta\cos^2\theta - \frac{1}{8}\cos 4\theta\, d\theta$$

$$= 2\left(\frac{5}{8}\theta + \frac{1}{4}\sin 2\theta - \frac{2}{3}\cos^3\theta - \frac{1}{32}\sin 4\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)$$



$$= 2\left(\frac{5}{16}\pi + \frac{5}{16}\pi + 0 - 0 - 0 + 0 - 0 + 0\right)$$

$$= \frac{5}{4}\pi.$$

24) $(x-1)^2+y^2=1$

$(x-1)^2=1-y^2 \qquad x=1\pm\sqrt{1-y^2}$

$$md_{x\text{-axis}} = 2\int_0^1\int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} x\cdot x^2\,dx\,dy$$

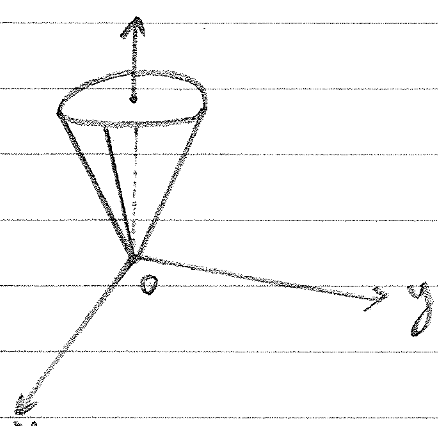
$$=\frac{2}{4}\int_0^1 x^4\Big|_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}}\,dy$$

$$=\frac{1}{2}\int_0^1 1+4\sqrt{1-y^2}+6(1-y^2)+4(1-y^2)^{\frac{3}{2}}+(1-y^2)^2$$
$$-\left[1-4\sqrt{1-y^2}+6(1-y^2)-4(1-y^2)^{\frac{3}{2}}+(1-y^2)^2\right]dy$$

$$=\frac{1}{2}\int_0^1 8\sqrt{1-y^2}+8(1-y^2)^{\frac{3}{2}}\,dy$$

$$=4\int_0^1\sqrt{1-y^2}+(1-y^2)^{\frac{3}{2}}\,dy$$

HW
25. Calculate the area of the part of the cone $z^2=x^2+y^2$, lying in the region of space defined by $x\geq 0$, $y\geq 0$, $z<1$.



$z=\sqrt{x^2+y^2} \quad (x\geq 0, y\geq 0)$.

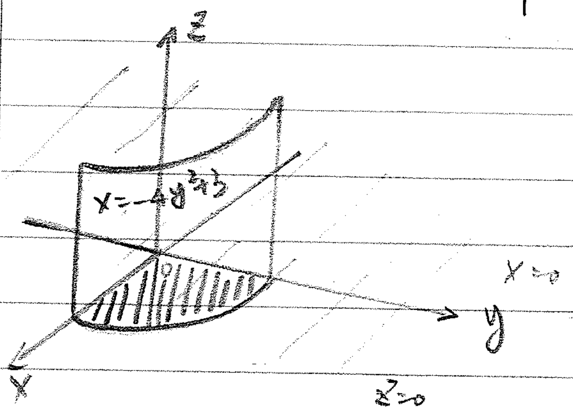
$z=f(x,y)$

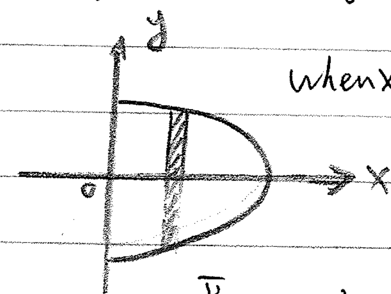
Surface Area of a Graph

$$\text{Area}=\iint_D dA=\iint_D\sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\,dx\,dy$$
$$=\iint_D\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

$$\iint\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dx\,dy=\sqrt{2}\iint dx\,dy=\frac{\sqrt{2}}{4}\pi \quad \left(\text{Noticing that } \iint dx\,dy=\frac{1}{4}\pi\right)$$

9. Find the volume of the region bounded by the planes $x=0$, $z=0$, and the surfaces $x=-4y^2+3$, and $z=x^3y$





when $x=0$, $y^2=\frac{3}{4}$

$y=\pm\frac{\sqrt{3}}{2}$

$$V=\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}\int_{0}^{-4y^2+3} x^3y\,dx\,dy$$