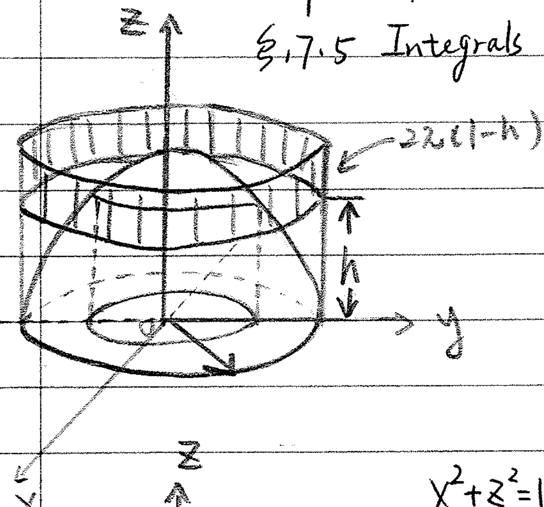
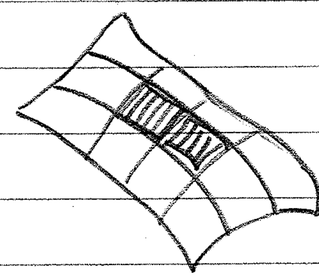
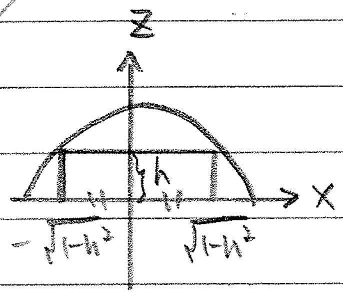
11.26. Chapter 17 Multiple Integration

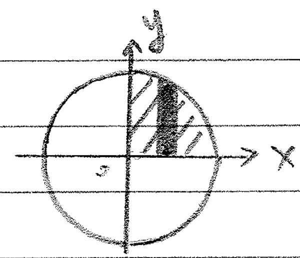
§17.5 Integrals in Polar, Cylindrical, and Spherical Coordinates







$x^2+z^2=1$. $x^2=1-z^2$ $\Rightarrow$ $x=\pm\sqrt{1-h^2}$



$y=\sqrt{1-h^2-x^2}$

$$x^2+y^2+z^2=1 \rightarrow z^2=1-x^2-y^2$$

$$z=f(x,y)=\sqrt{1-x^2-y^2}$$

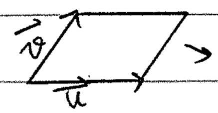
$$f_x=\frac{1}{2}(1-x^2-y^2)^{-\frac{1}{2}}\cdot(-2x)=-\frac{x}{\sqrt{1-x^2-y^2}}$$

$$f_y=-\frac{y}{\sqrt{1-x^2-y^2}}$$

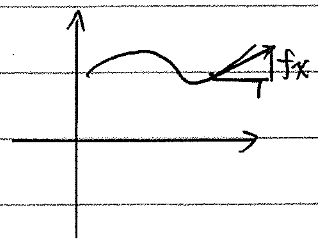
$$\iint_D \sqrt{1+f_x^2+f_y^2}\,dx\,dy=\iint_D\sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}\,dx\,dy$$

$$=\iint_D \frac{1}{\sqrt{1-x^2-y^2}}\,dx\,dy$$

$$=4\int_0^{\sqrt{1-h^2}}\int_0^{\sqrt{1-h^2-x^2}}\frac{1}{\sqrt{1-x^2-y^2}}\,dy\,dx.$$



area $= \|\vec{u} \times \vec{v}\|$.



$i + f_x k$

$j + f_y k$.

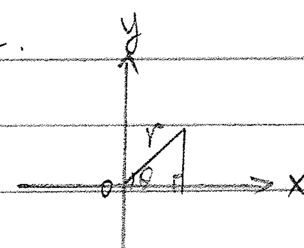
$$\begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x i - f_y j + k$$

~~Length~~: $\sqrt{f_x^2 + f_y^2 + 1}$

Thus,

1> Surface area $= \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$

2>. Change to Polar Coordinate.



$x = r\cos\theta$. $\quad x(r, \theta)$

$y = r\sin\theta$. $\quad y(r, \theta)$

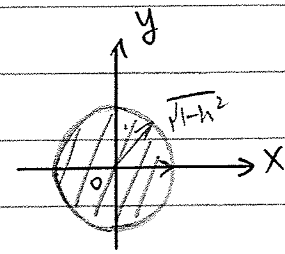
$$\iint_D f(x, y)\, dx\, dy = \iint_D f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

$$= \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

det: $r\cos^2\theta + r\sin^2\theta = r$.

$$4\int_0^{\sqrt{1-h^2}}\int_0^{\sqrt{1-h^2-x^2}} \frac{1}{\sqrt{1-x^2-y^2}}\,dy\,dx$$



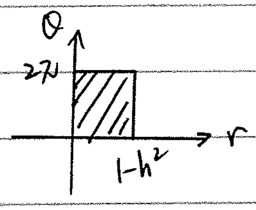
$$=\int_0^{2\pi}\int_0^{\sqrt{1-h^2}} \frac{1}{\sqrt{1-r^2\cos^2\theta-r^2\sin^2\theta}}\, r\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\sqrt{1-h^2}} \frac{r}{\sqrt{1-r^2}}\,dr\,d\theta$$

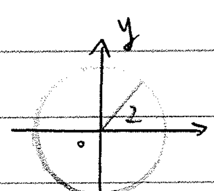
Assume $1-r^2=u$.

$du=-2r\,dr$.

$$=-\frac{1}{2}\int_0^{2\pi} 2u^{\frac{1}{2}}\Big|_0^{\sqrt{1-h^2}}\,d\theta$$



Ex. HW ① Evaluate $\iint_D (x^2+y^2)^{3/2}\,dx\,dy$, where D is the disk $x^2+y^2\le 4$.

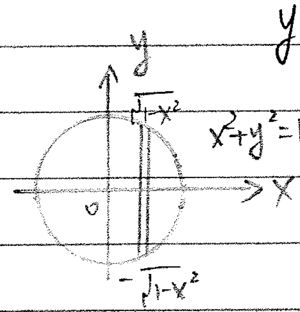


$$\iint_D (x^2+y^2)^{3/2}\,dx\,dy = \int_0^{2\pi}\int_0^2 r^3\cdot r\,dr\,d\theta$$

HW ⑥ Find $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin(x^2+y^2)\,dy\,dx$ by converting to polar coordinates.

$y=\pm\sqrt{1-x^2} \Rightarrow x^2+y^2=1$

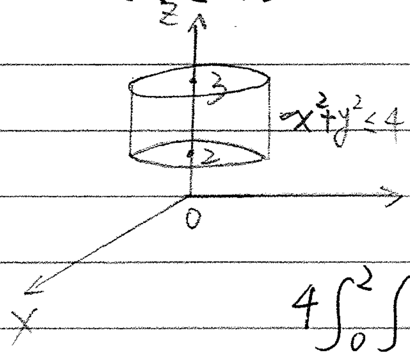
$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}$$

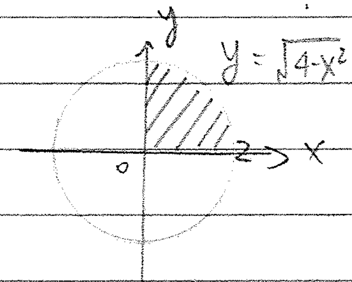


$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin(x^2+y^2)\,dy\,dx$$

$$=\frac{1}{2}\int_0^{2\pi}\int_0^1 \sin(r^2)\cdot 2r\,dr\,d\theta$$

HW ⑦ Integrate $ze^{x^2+y^2}$ over the cylinder $x^2+y^2 \le 4$, $2 \le z \le 3$





$$4\int_0^2\int_0^{\sqrt{4-x^2}}\int_2^3 ze^{x^2+y^2}\,dx\,dy\,dz$$

Cylindrical Coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

det: $r\cos^2\theta - 0 + r\cdot\sin^2\theta - 0 + 0 - 0 = r$

$$\iiint_W ze^{x^2+y^2}\,dx\,dy\,dz = \frac{1}{2}\int_2^3\int_0^{2\pi}\int_0^2 z\cdot e^{r^2}\cdot 2r\,dr\,d\theta\,dz$$

HW ⑨ Evaluate $\iiint_W \frac{dx\,dy\,dz}{\sqrt{1+x^2+y^2+z^2}} = A$

Spherical Coordinates

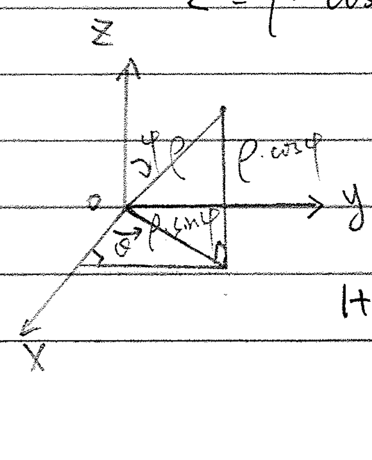
$$\begin{cases} x = \rho\cdot\sin\varphi\cdot\cos\theta \\ y = \rho\cdot\sin\varphi\,\sin\theta \\ z = \rho\cdot\cos\varphi. \end{cases}$$

$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| = \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix}$$

$= |\sin\varphi\cos\theta(-\rho^2\sin^2\varphi\cos\theta) + \rho\sin\varphi\sin\theta(-\rho\sin^2\varphi\sin\theta - \rho\cos^2\varphi\sin\theta) + \rho\cos\varphi\cos\theta(-\rho\sin\varphi\cos\varphi\cos\theta$

$= \rho^2\sin^3\varphi\cos^2\theta + \rho^2\sin\varphi\cos^2\varphi\cos^2\theta + \rho^2\sin\varphi\sin^2\theta$

$= \rho^2\cdot\sin\varphi$



$1+x^2+y^2+z^2 = 1+\rho^2$

$$A = \int_0^{\pi}\int_0^{2\pi}\int_0^1 \frac{\rho^2\sin\varphi}{\sqrt{1+\rho^2}}\,d\rho\,d\theta\,d\varphi$$

11.28 Chapter 17 Multiple Integration §17.6 & §17.5

§17.6 Applications of Triple Integrals

Formulas:

Volumn: $\iiint_W dx\,dy\,dz$

Mass: $\iiint_W \rho(x,y,z)\,dx\,dy\,dz$

Center of mass $= (\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{\iiint_W x\rho(x,y,z)\,dx\,dy\,dz}{\text{mass}}$$

$$\bar{y} = \frac{\iiint_W y\rho(x,y,z)\,dx\,dy\,dz}{\text{mass}}$$

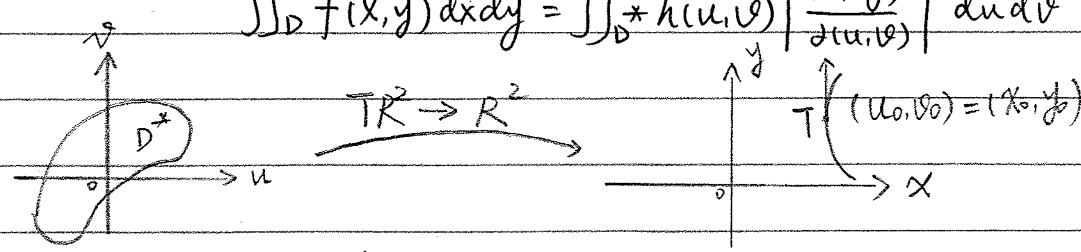
$$\bar{z} = \frac{\iiint_W z\rho(x,y,z)\,dx\,dy\,dz}{\text{mass}}$$

The average value of a function f on a region W:

$$\frac{\iiint_W f(x,y,z)\,dx\,dy\,dz}{\iiint_W dx\,dy\,dz}$$

§17.5 Verify:

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} h(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

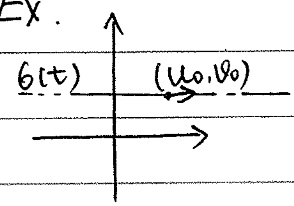


2×2 2×1

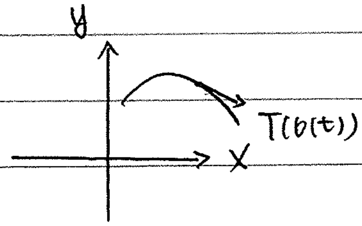
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 3b \\ 3d \end{pmatrix} \qquad v = (2,1) = 2i + j$$

$$\frac{\partial(x,y)}{\partial(u,v)}\begin{pmatrix}w_1\\ w_2\end{pmatrix} = \begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix} = \begin{pmatrix}\frac{\partial x}{\partial u}\cdot w_1+\frac{\partial x}{\partial v}\cdot w_2\\ \frac{\partial y}{\partial u}\cdot w_1+\frac{\partial y}{\partial v}\cdot w_2\end{pmatrix}$$

Ex.



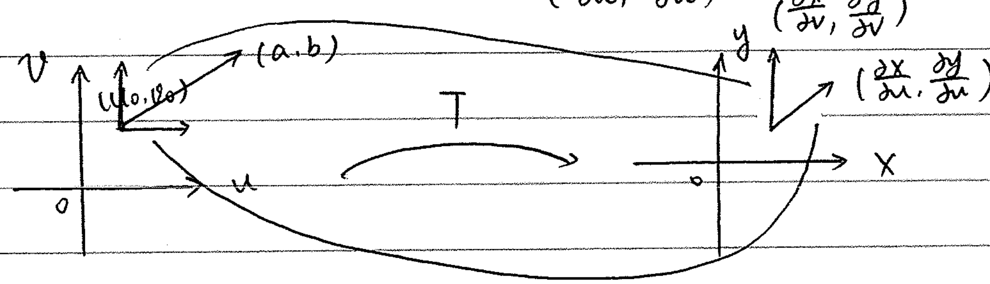
$$\sigma(t)=(u_0,v_0)+t(1,0)=(u_0+t,v_0)$$



$$T(\sigma(0)) = T(u_0,v_0) = (x_0,y_0) = (x(u_0+t,v_0),\ y(u_0+t,v_0)$$

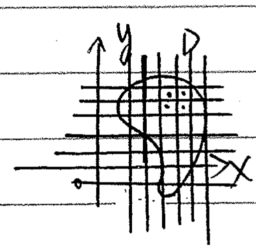
$$T'(\sigma(t)) = \left(\frac{\partial x}{\partial u}\times 1+\frac{\partial x}{\partial v}\times 0,\ \frac{\partial y}{\partial u}\times 1+\frac{\partial y}{\partial v}\times 0\right)$$

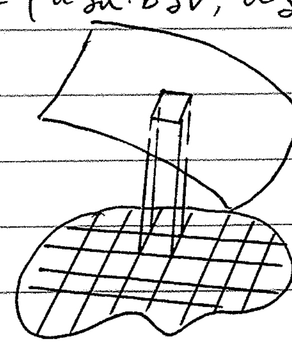
$$= \left(\frac{\partial x}{\partial u},\ \frac{\partial y}{\partial u}\right)$$

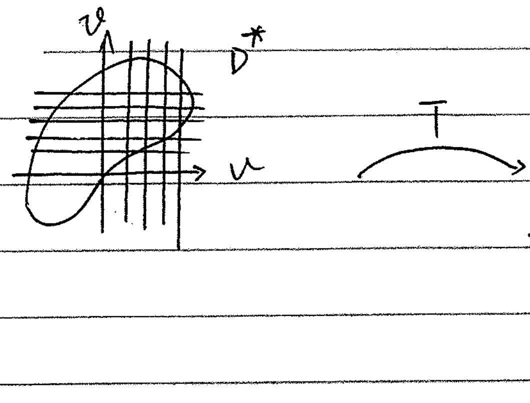


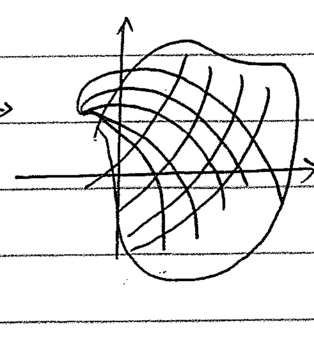
$$\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = a\left(\frac{\partial x}{\partial u},\frac{\partial y}{\partial u}\right)+b\left(\frac{\partial x}{\partial v},\frac{\partial y}{\partial v}\right) = \left(a\frac{\partial x}{\partial u}+b\frac{\partial x}{\partial v},\ a\frac{\partial y}{\partial u}+b\frac{\partial y}{\partial v}\right)$$

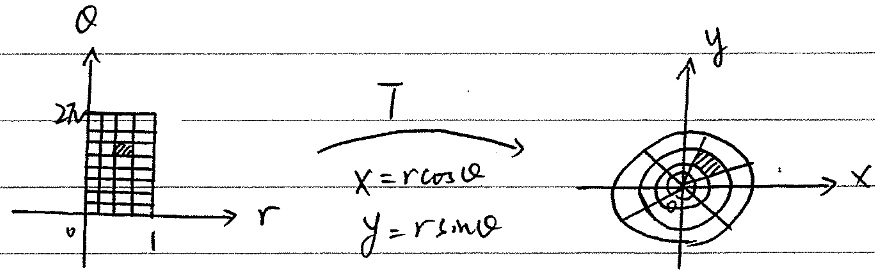
$$\iint_D f(x,y)\,dx\,dy$$

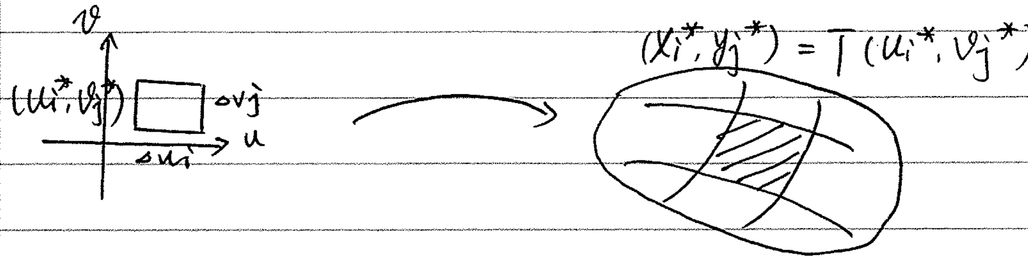












$$\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\begin{pmatrix}\Delta u_i\\ 0\end{pmatrix} + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)\cdot\begin{pmatrix}0\\ \Delta v_j\end{pmatrix}$$

$$= \frac{\partial(x,y)}{\partial(u,v)}\cdot\begin{pmatrix}\Delta u_i & 0\\ 0 & \Delta v_j\end{pmatrix}$$

$$\det:\ \left|\frac{\partial(x,y)}{\partial(u,v)}\Delta u_i \Delta v_j\right|$$

$$\iint_D f(x,y)\,dx\,dy \approx \sum f(x_i^*, y_j^*)\left(\text{area of } T \text{ of } \square_{\Delta u_i}\Delta v_j\right)$$

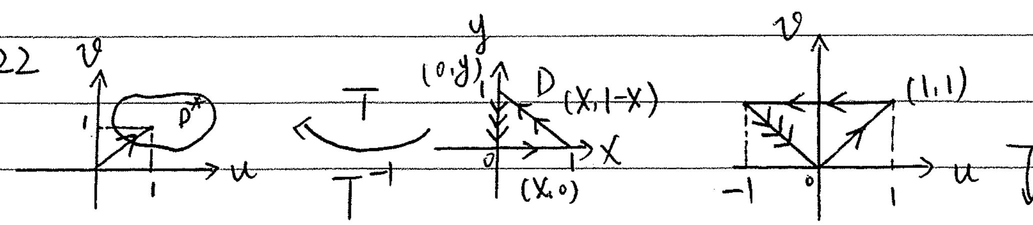
$$\approx \sum f(x(u_i^*, v_j^*), y(u_i^*, v_j^*))$$

$\cdot$ (area of parallelgram

$$= \iint_{D^*} h(u,v)$$

$$\left(\det\left(\frac{\partial(x,y)}{\partial(u,v)}\right)\right) du\,dv.$$

HW22



$u_2 = x-(1-x) = 2x-1$

$$T(u,v) = (x,y) = \left(\frac{u+v}{2}, \frac{v-u}{2}\right)$$

$v_2 = x+(1-x) = 1$

$u_3 = -y$

$v_3 = y$

HW22 Let D be the region bounded by $x+y=1$, $x=0$, $y=0$. Show:

$$\iint_D \cos\left(\frac{x-y}{x+y}\right) dx\,dy = \frac{\sin 1}{2}$$

& graph D on an xy plane and a uv plane, with $u=x-y$, $v=x+y$.

$$\iint_D \cos\left(\frac{x-y}{x+y}\right) dx\,dy$$

$$\begin{cases} u = x-y \\ v = x+y \end{cases} \Rightarrow \begin{cases} x = \frac{u+v}{2} \\ y = \frac{v-u}{2} \end{cases}$$

$$T^{-1}(x,y) = (u,v) = (x-y, x+y)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\det = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\iint_{D^*} \cos\left(\frac{u}{v}\right) \cdot \frac{1}{2}\, du\, dv$$

$$= \frac{1}{2} \cdot 2 \int_0^1 \int_0^v \cos\left(\frac{u}{v}\right) du\, dv$$

$$= \int_0^1 v \sin\left(\frac{u}{v}\right) \Big|_0^v dv$$

$$= \int_0^1 v \sin 1\, dv$$

$$= \frac{\sin 1}{2}$$

11.30 Chapter 18 Vector Analysis

§18.1 Line Integrals

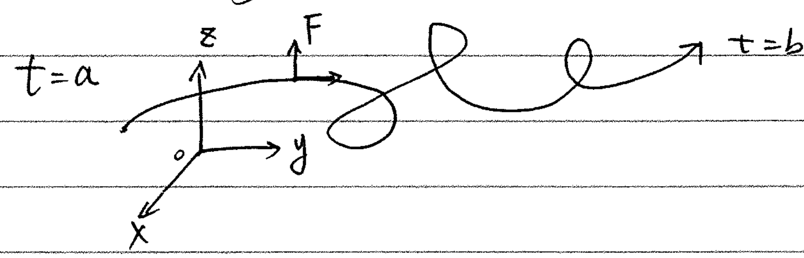
$$K = \frac{1}{2} m |\vec{v}|^2 = \frac{1}{2} m \vec{v}\cdot\vec{v}$$

$$\frac{dk}{dt} = \frac{1}{2} m (\vec{a}\cdot\vec{v} + \vec{v}\cdot\vec{a})$$

$$= m\cdot\vec{a}\cdot\vec{v}$$

$$F = ma = \vec{F}\cdot\vec{v}$$

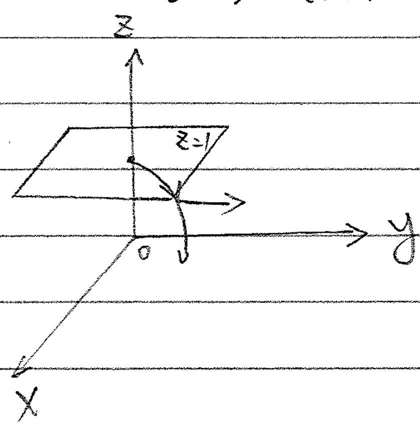
$$\Rightarrow \text{work} = \int_a^b \vec{F}\cdot\vec{v}\,dt = \int_a^b F(\sigma(t))\cdot\sigma'(t)\,dt$$



Ex. HW1. Calculate the work which is done by the force field $\Phi(x,y,z) = xi + yj$ when a particle is moved along the path $(3t^2, t, 1)$, $0 \le t \le 1$.

$$\Phi(x,y,z) = xi + yj = (x, y, 0)$$

$$\sigma(t) = (3t^2, t, 1) \qquad x = 3t^2,\ y = t \Rightarrow x = 3y^2$$



$$\int_0^1 \Phi(\sigma(t))\cdot\sigma'(t)\,dt \qquad \sigma'(t) = (6t, 1, 0)$$

$$= \int_0^1 (3t^2, t, 0)(6t, 1, 0)\,dt$$

$$= \int_0^1 18t^3 + t\,dt$$

$$= \frac{18}{4}t^4 + \frac{1}{2}t^2 \Big|_0^1$$

$$= \frac{9}{2} + \frac{1}{2} = 5$$

HW7 Show that if a particle is moved along a closed curve $(\cos t, \sin t, 0)$, $0 \le t \le 2\pi$, then the force field $yi - xj + k$ does a nonzero amount of work on the particle. How much is the work?

$\sigma(t) = (\cos t, \sin t, 0) \quad 0 \le t \le 2\pi$

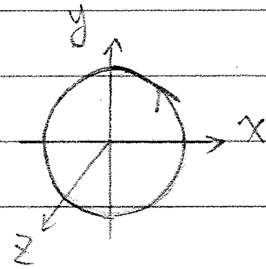
$\Phi(x, y, z) = (y, -x, 1)$

$\sigma'(t) = (-\sin t, \cos t, 0)$

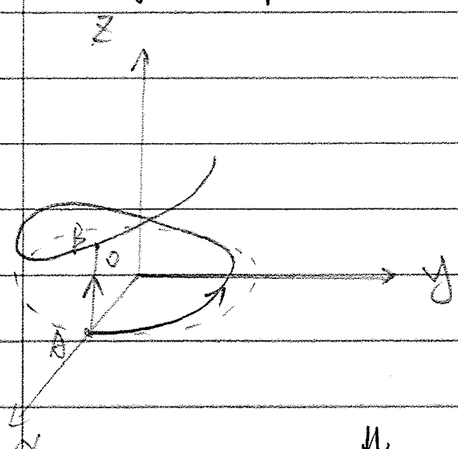
$\int_0^{2\pi} (\sin t, -\cos t, 1)(-\sin t, \cos t, 0)\, dt$

$= \int_0^{2\pi} -1\, dt$

$= -2\pi$



* If the path is a helix



$\sigma(t) = (\cos t, \sin t, \frac{t}{2\pi})$

$\sigma'(t) = (-\sin t, \cos t, \frac{1}{2\pi})$

$\int_0^{2\pi} (\sin t, -\cos t, 1)(-\sin t, \cos t, \frac{1}{2\pi})\, dt$

$= \int_0^{2\pi} \frac{1}{2\pi} - 1\, dt = 1 - 2\pi.$

$\Downarrow$

$\sigma(t) = (1, 0, t) \quad 0 \le t \le 1 \qquad \Phi = (y, -x, 1)$

$\sigma'(t) = (0, 0, 1)$

$\int_0^1 (0, -1, 1) \cdot (0, 0, 1)\, dt$

$= \int_0^1 dt = 1$

if we change 1 to t^3

$\gamma(t) = (1, 0, t^3) \quad 0 \le t \le 1$

$\gamma'(t) = (0, 0, 3t^2)$

$\int_0^1 (0, -1, 1) \cdot (0, 0, 3t^2)\, dt$

$= \int_0^1 3t^2\, dt$

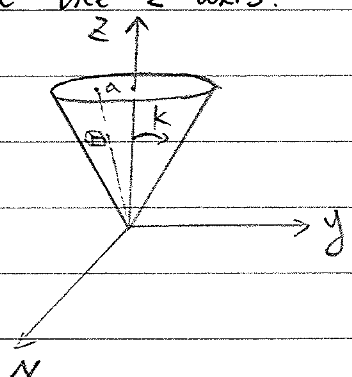
$= t^3 \Big|_0^1$

$= 1$

We get the same answer

§7.6

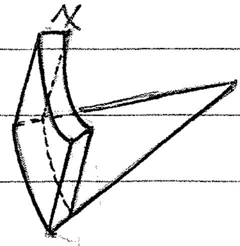
HW11. A solid with constant density is bounded above by the plane $z = a$ and below by the cone described in spherical coordinates by $\phi = K$, where K is a constant $0 < K < \pi/2$. Set up an integral for its moment of inertia about the z axis.

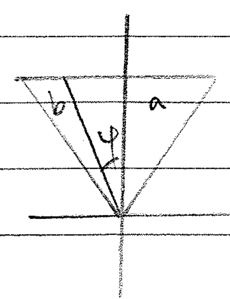


$0 < K < \frac{\pi}{2}$

$$\int_0^{2\pi} \int_0^{K} \int_0^{\frac{a}{\cos\varphi}} \delta\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$\Rightarrow \delta \int_0^{2\pi} \int_0^{K} \frac{\rho^3}{3} \Big|_0^{\frac{a}{\cos\varphi}} \sin\varphi\, d\varphi\, d\theta$$





$\frac{a}{b} = \cos\varphi$

$b = \frac{a}{\cos\varphi}$

12.3

Chapter 18 Vector Analysis

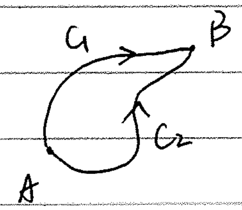
§18.2 Path Independence

Conservative < Definition >

A vector field is conservative if $\int_{C_1} \Phi(r)\,dr = \int_{C_2} \Phi(r)\,dr$

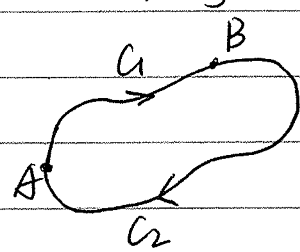
the line integrals of Φ being path independent.

(it doesn't depend on the path; it only depends on the end point of path)



$$\int_{C_1} \Phi(r)\,dr = \int_{C_2} \Phi(r)\,dr$$

< Theorem > A vector field is conservative if & only if $\int \Phi(r)\,dr$ around any loop is zero.



$$\int \Phi(r)\,dr$$

$$= \int_{C_1} \Phi(r)\,dr + \int_{C_2} \Phi(r)\,dr$$

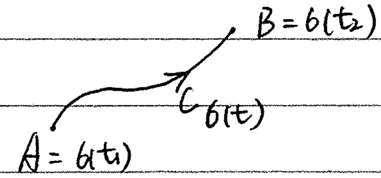
$$= \int_{C_1} \Phi(r)\,dr - \int_{-C_2} \Phi(r)\,dr$$

$$= 0$$

< Theorem > A vector field is conservative if & only if $\Phi = \nabla f$. then for any curve C from A to B.

$$\int_C \Phi(r)\,dr = f(B) - f(A)$$

<Proof> Suppose $\Phi = \nabla f$



$$\int_{t_1}^{t_2} \Phi(\sigma(t)) \cdot \sigma'(t)\, dt$$

$$= \int_{t_1}^{t_2} \nabla f(\sigma(t)) \cdot \sigma'(t)\, dt$$

$$= \int_{t_1}^{t_2} \frac{d\, f(\sigma(t))}{dt}\, dt \quad \text{— by Chain Rule}$$

$$= f(\sigma(t)) \Big|_{t_1}^{t_2}$$

$$= f(B) - f(A)$$

Suppose Φ is conservative

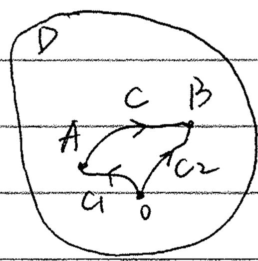
Pick a point 0 in the domain of Φ and define $f(0) = 0$

Now given A, define $f(A) = \int_C \Phi(r)\, dr$

I need to show that $\Phi = \nabla f$

Let's show first, that:

$$\int_C \Phi(r)\, dr = \int_C \nabla f(r) \cdot dr = f(B) - f(A)$$



$$\because \int_{C_1} \Phi(r)\, dr + \int_C \Phi(r)\, dr = \int_{C_2} \Phi(r)\, dr$$

$$\Rightarrow f(A) + \int_C \Phi(r)\, dr = f(B)$$

$$\therefore \int_C \Phi(r)\, dr = f(B) - f(A)$$

So, $\int_C (\Phi(r) = \nabla f(r))\, dr = 0$ for every path C.

<Theorem> Every Conservative Vector Field is a Gradient

If $\Phi(x,y)$ is a vector field defined in a region D, and Φ is conservative, then there is a function field f defined on D such that $\Phi = \nabla f$.

If $\int_C \varphi(r)\, dr = 0$ for every path c,

then $\varphi(r) = 0$ for all

Ex. $\Phi(x,y) = \varphi_1(x,y)i + \varphi_2(x,y)j$

If it is a ∇f of some function?

$$\begin{tikzcd}
 & f(x,y) \arrow[dl, "\frac{\partial f}{\partial x}"'] \arrow[dr, "\frac{\partial f}{\partial y}"] & \\
\varphi_1(x,y)i \arrow[dr, bend right, "\frac{\partial f}{\partial y}"'] & & \varphi_2(x,y)j \arrow[dl, "\frac{\partial f}{\partial x}"] \\
 & \text{SAME.} &
\end{tikzcd}$$

According to the mixed partial theorem,

$\Phi = \nabla f$ for some f

if & only if:

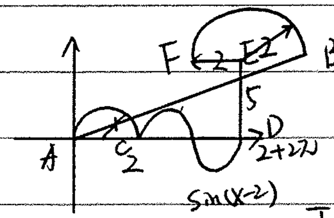
$$\frac{\partial \varphi_1}{\partial y} = \frac{\partial \varphi_2}{\partial x}$$

<Theorem> The Cross-Derivative Test

A vector field $\Phi(x,y) = a(x,y)i + b(x,y)j$ defined on the whole plane is conservative if & only if:

$$a_y = b_x.$$

* Gravity field → conservative



Line Integral ACDEFB

$= $ Line Integral AB

$\Phi(x,y) = \nabla f \qquad = f(B) - f(A)$

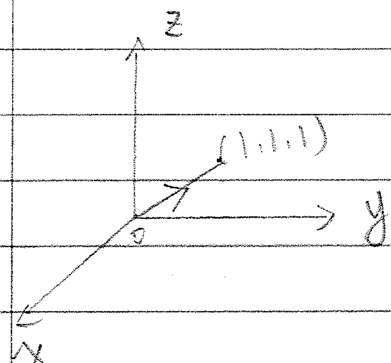
§18.1 HW 21. 17. 19

21. Let $\Phi(x,y,z) = x^2 i - xy j + k$

along the straight line joining $(0,0,0)$ to $(1,1,1)$

Find the line integral.

Parametrize the line: $(0,0,0) + t(1,1,1) = (t,t,t) \quad t \in [0,1]$



$\sigma(t) = (t,t,t) \qquad \sigma'(t) = (1,1,1)$

$\int_0^1 \Phi(\sigma(t)) \cdot \sigma'(t)\, dt$

$= \int_0^1 (t^2, -t^2, 1)(1,1,1)\, dt$

$= t\Big|_0^1 = 1$

17. $\sigma(t) = (\sin t, t^2, t) \quad t \in [0, 2\pi] \qquad \sigma'(t) = (\cos t, 2t, 1)$

$\Phi(x,y,z) = \sin z\, i + \cos\sqrt{y}\, j + x^3 k$

19. $\sigma(t) = ((1+t^2)^2, 1, t) \quad t \in [0,1] \qquad \sigma'(t) = (2(1+t^2)\cdot 2t, 0, 1)$

$\Phi(x,y,z) = \left(\frac{1}{z^2+1}, x(1+y^2), e^y\right)$

$\int_0^1 \Phi(\sigma(t)) \cdot \sigma'(t)\, dt = \int_0^1 \left(\frac{1}{t^2+1}, 2(1+t^2)^2, e\right)(2(1+t^2)\cdot 2t, 0, 1)\, dt$

$= \int_0^1 4t + e\, dt = 2t^2\Big|_0^1 + et\Big|_0^1 = 2 + e$