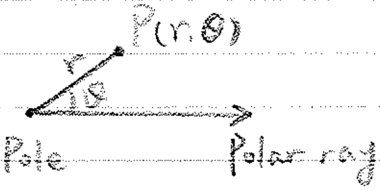
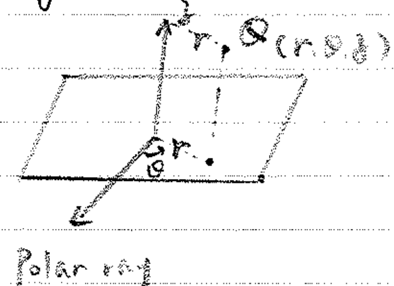
9.28 §14.5 Cylindrical and Spherical Coordinates

1. Cylindrical Coordinates

① polar curve V.S. cylindrical coordinates





②

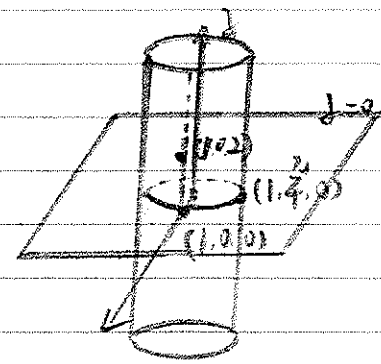
$r = \sqrt{x^2+y^2}$

$\theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x \geq 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0 \end{cases} \quad \longleftrightarrow \quad \begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \\ z = z \end{cases}$

(r, θ, z) cylindrical V.S. (x, y, z) Cartesian coordinates

Ex1 sketch the surface whose equation in cylindrical coordinates is $r=1$.

$r=1$



$x^2+y^2=1$

transform Cartesian coordinates to cylindrical coordinates.

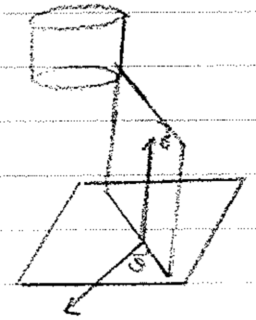
Ex2 $x^2+y^2=1$ substituting $\begin{cases} x = r\cdot\cos\theta \\ y = r\cdot\sin\theta \end{cases}$ into $x^2+y^2=1$

$\Rightarrow r^2\cdot\cos^2\theta + r^2\cdot\sin^2\theta = 1 \Rightarrow r^2\cdot(\cos^2\theta+\sin^2\theta) = 1 \Rightarrow r^2 = 1$

$\Rightarrow r = \pm 1 \Leftrightarrow r = 1$

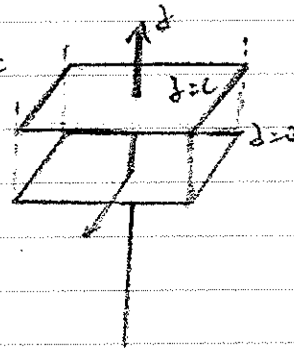
Ex3. sketch the surface of equations in cylindrical coordinates

① r is constant

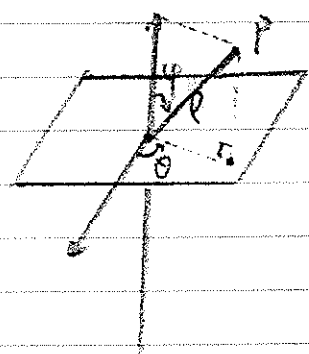


② θ is constant $\Rightarrow$ a plane contains z-axis

③ z is constant



2. Spherical Coordinates

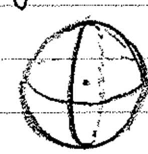


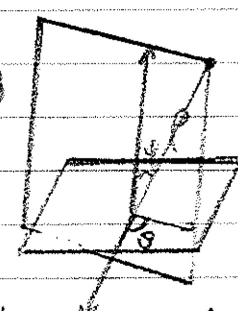
$x^2+y^2+z^2=r^2$

Ex1.

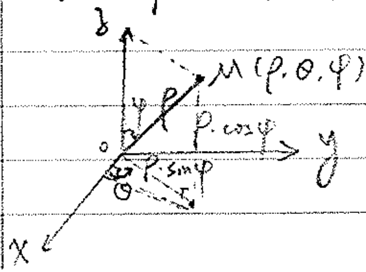
ρ is constant. $\to$ sphere

θ is constant $\to$ plane





transform cartesian coordinates to spherical coordinates



$$\begin{cases} x = \rho\cdot\sin\varphi\cdot\cos\theta \\ y = \rho\cdot\sin\varphi\cdot\sin\theta \\ z = \rho\cdot\cos\varphi \end{cases} \Leftrightarrow$$

$\rho \geq 0,\ \theta \in [-\pi, \pi]$

$\varphi \in [0, \pi]$

$\rho = \sqrt{x^2+y^2+z^2}$

$$\theta = \begin{cases} \tan^{-1}(y/x), & x \geq 0 \\ \tan^{-1}(y/x)+\pi, & x < 0 \end{cases}$$

$$\varphi = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$$

Ex2

transform cartesian coordinates to spherical coordinates

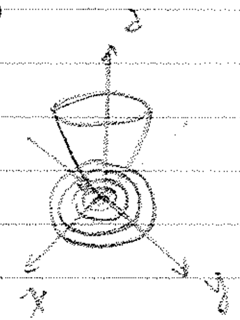
$z = x^2+y^2$

substituting $\begin{cases} z = \rho \cdot \cos\varphi \\ x = \rho \cdot \sin\varphi \cdot \cos\theta \\ y = \rho \cdot \sin\varphi \cdot \sin\theta \end{cases}$ into $z = x^2+y^2$

$$\Rightarrow \quad \rho \cdot \cos\varphi = \rho^2 \cdot \sin^2\varphi \cdot \cos^2\theta + \rho^2 \cdot \sin^2\varphi \cdot \sin^2\theta$$

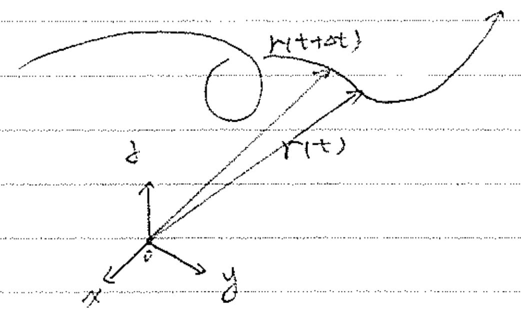
$$= \rho^2 \cdot \sin^2\varphi \, (\sin^2\theta + \cos^2\theta)$$

$$= \rho^2 \cdot \sin^2\varphi$$



$$\Rightarrow \quad \rho = \frac{\cos\varphi}{\sin^2\varphi}$$

10.01 §14.6 Curves in Space



$r(t) = (x(t), y(t), z(t))$

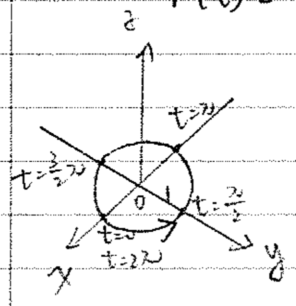
$r(t)$ is the position vector of the particle

$$r'(t) = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t} = \lim_{\Delta t \to 0} \left(\frac{x(t+\Delta t) - x(t)}{\Delta t}, \frac{y(t+\Delta t) - y(t)}{\Delta t}, \frac{z(t+\Delta t) - z(t)}{\Delta t} \right)$$

$$= (x'(t), y'(t), z'(t))$$

Ex1 $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = 0$

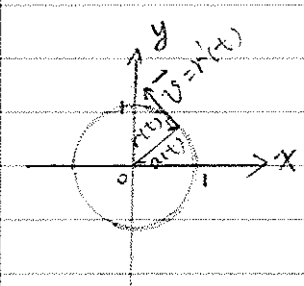
$r(t) = (\cos t, \sin t, 0)$



$\vec{v} = r'(t) = (-\sin t, \cos t, 0)$

$\because r(t) \cdot r'(t) = -\sin t \cos t + \sin t \cdot \cos t + 0 = 0$

$\therefore \vec{v} \perp r(t)$

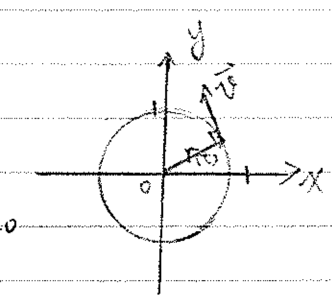


$\vec{a} = r''(t) = (-\cos t, -\sin t, 0)$

$= -r(t)$

speed $= \|\vec{v}\| = \sqrt{\sin^2 t + \cos^2 t + 0} = 1$

Ex2. $r(t) = (\cos(t^2), \sin(t^2), 0)$



$\vec{v} = r'(t) = (-2t \sin t^2, 2t \cdot \cos t^2, 0)$

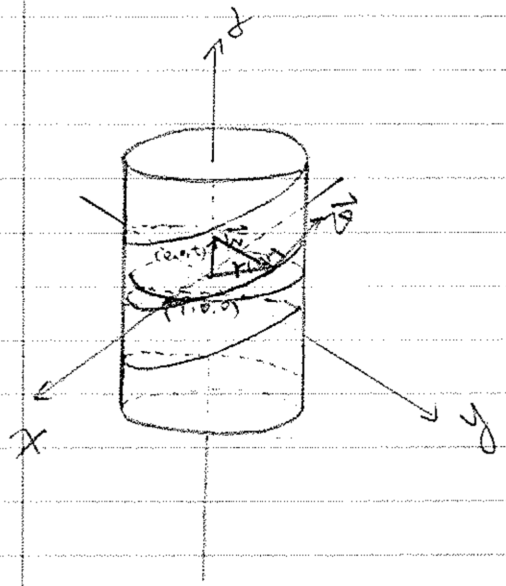
$r'(t) \cdot r(t) = -2t \sin t^2 \cos t^2 + 2t \sin t^2 \cos t^2 + 0 = 0$

$\Rightarrow \vec{v} \perp r(t)$

Speed $= \|\vec{v}\| = \sqrt{4t^2 \sin^2 t^2 + 4t^2 \cos^2 t^2} = \sqrt{4t^2} = 2|t|$

$r''(t) = \vec{a} = (-2\sin t^2 - 4t^2 \cos t^2, 2\cos t^2 - 4t^2 \sin t^2, 0)$

Ex3. $r(t) = (\cos t, \sin t, t)$ Rt-handed helix



$t = 0,\ (1, 0, 0)$

$\vec{v} = r'(t) = (-\sin t, \cos t, 1)$

$r(t) \cdot r'(t) = \cos t \cdot (-\sin t) + \sin t \cdot \cos t + t$

$= t$

$\Rightarrow r(t) \not\perp r'(t)$

$\vec{w} = r(t) - (0, 0, t)$

$= (\cos t, \sin t, 0)$

$\Rightarrow \vec{w} \perp r'(t)$

$\vec{a} = r''(t) = (-\cos t, -\sin t, 0) = -\vec{w}$

$\Rightarrow \vec{v} \perp \vec{a}\ \ (r'(t) \perp r''(t))$

Speed $= \|\vec{v}\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

* Differentiation Rules for Vector Functions

Sum Rule: $\dfrac{d\ r(t) + s(t)}{dt} = \dfrac{dr}{dt} + \dfrac{ds}{dt}$

Dot Product Rule: $\dfrac{d\ r(t) \cdot s(t)}{dt} = r(t) \cdot \dfrac{ds}{dt} + s(t) \dfrac{dr}{dt}$

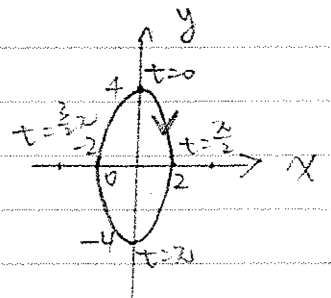
Cross Product Rule: $\dfrac{d\ r(t) \times s(t)}{dt} = r'(t) \times s(t) + r(t) \times s'(t)$

* Chain Rule: $\dfrac{d}{dt}[r(s(t))] = s'(t) \cdot r'(s(t))$

Ex4. $x = 2\sin t,\ y = 4\cos t,\ z = 0$, sketch the graph.

$$\Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

$$\Rightarrow \frac{x^2}{4} + \frac{y^2}{16} = 1, \quad \text{ellipse} \begin{cases} a = 2 \\ b = 4 \end{cases}$$

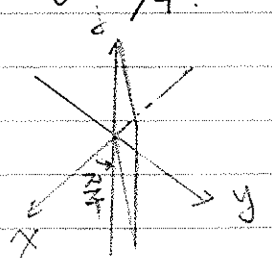


$\Rightarrow$ $x = a\sin t,\ y = b\cos t,\ z = 0$

$\Rightarrow$ ellipse. (if $a \neq b$).

$a = b$, circle.

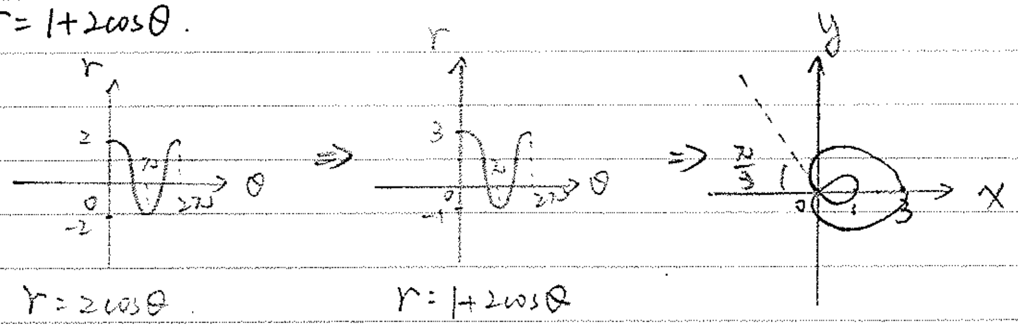
HW 35. Describe the surface given in spherical coordinates by $\theta = \pi/4$.

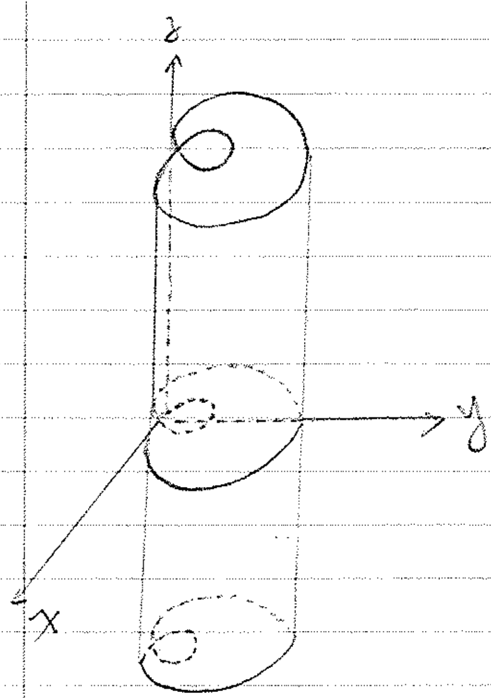


$*$ in spherical coordinates $\Rightarrow \rho \geq 0$.

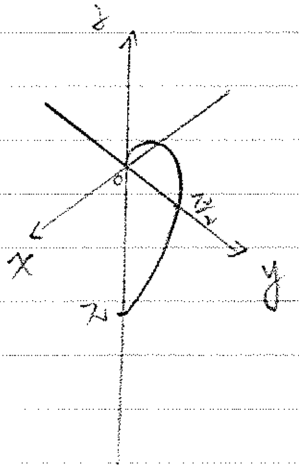
the vertical half plane with positive y-coordinates and making a 45° angle with the x-y plane.

HW 13. Sketch the surface described in cylindrical coordinates by $r = 1 + 2\cos\theta$.



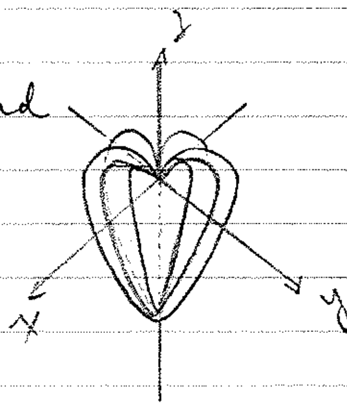


HW 36. Describe the surface given in spherical coordinates by $\rho = \phi$



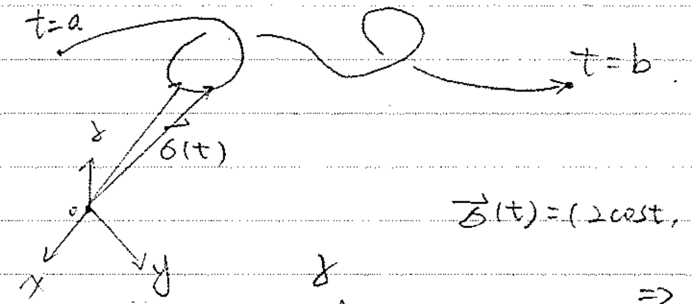
Spin around

$\Rightarrow$



10.03 §14.7 The Geometry and Physics of Space Curves

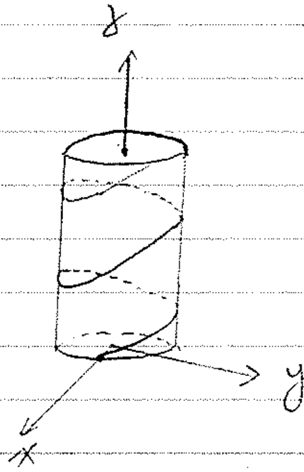
1). Distance travelled by a particle



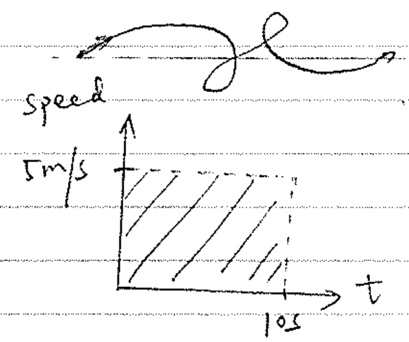
$\vec{\sigma}(t) = (2\cos t, 2\sin t, t)$

$\Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ & $z = t$

Graph:

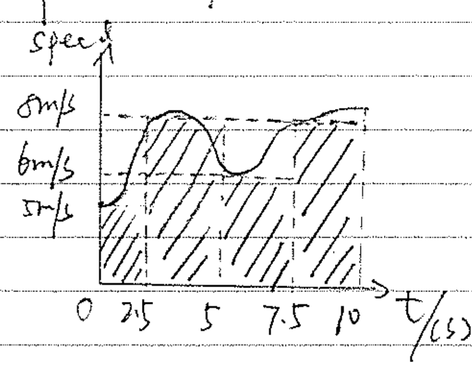


① if the speed is constant



$D = 5m/s \times 10s = 50m$

② if the speed is variable



$\boxed{* \text{ distance travelled by a particle} = \int_a^b \text{speed} \cdot dt}$

$D = 5m/s \times 2.5s + 8m/s \times 2.5s + 6m/s \times 2.5s + 8m/s \times 2.5s$

$= 12.5m + 20m + 15m + 20m$

$= 67.5m$ → By using Riemann Sums.

Ex 1. Find the arc length of the given curve on the specified interval

① curve: $y = x^2$

interval: $[0, 4]$

$\sigma(t) = (t, t^2)$

$\vec{v} = \sigma'(t) = (1, 2t)$

$\|\vec{v}\| = \text{speed} = \sqrt{1^2 + (2t)^2} = \sqrt{4t^2+1}$

$L = \int_0^4 \sqrt{4t^2+1} \cdot dt$

② curve: $\sigma(t) = (\sin 3t, \cos 3t, 2t^{3/2})$

interval: $[0, 1]$

$\vec{v} = \sigma'(t) = (3\cos 3t, -3\sin 3t, 3t^{\frac{1}{2}})$

$\text{speed} = \|\vec{v}\| = \sqrt{9\cos^2 3t + 9\sin^2 3t + 9t}$

$= \sqrt{9+9t}$

$= 3\sqrt{1+t}$

$L = \int_0^1 3\sqrt{1+t} \cdot dt = 3\int_0^1 \sqrt{1+t} \cdot dt = 3 \cdot \frac{(1+t)^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^1$

$\Rightarrow 2 \cdot 2^{3/2} - 2 = 4\sqrt{2} - 2.$

③ Curve: $x = a \cdot \cos t,\ y = b \cdot \sin t$

interval: $[0, 2\pi]$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \sigma(t) = (a\cos t, b\sin t)$

$\sigma'(t) = (-a\sin t,\ b\cos t)$

speed $= \|\sigma'(t)\| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$

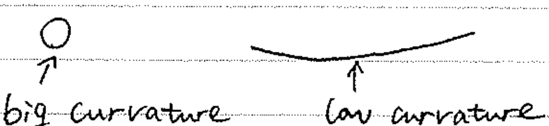
$L = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t} \cdot dt$

2). Curvature

I.

① a line has no curvature, i.e. 0.

② a circle has the curvature $= \frac{1}{r}$.



II. Osculating circle

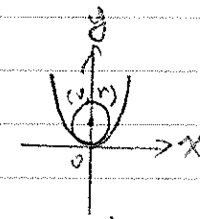
1) pick a point on the curve

2) find a circle pass the point

3) let the circle's second derivative equals to that of the point on the curve.

Ex1

Curve: $y = x^2$



1) pick a point $(0,0)$

2) the circle center at $(0,r)$, $x^2 + (y-r)^2 = r^2$

$y = x^2$

$y' = 2x$

$y'' = 2.$

$y - r = \pm\sqrt{r^2 - x^2}$ $\Rightarrow$ just pick the negative part

$y = r - \sqrt{r^2 - x^2}$

$y' = x(r^2 - x^2)^{-\frac{1}{2}}$

$y'' = \frac{r}{r^2} = \frac{1}{r} = 2.$ $\Rightarrow r = \frac{1}{2}$

$\Uparrow$

substituting $x = 0$ into y''

* General formula of curvature

$$\kappa = \frac{\| \vec{v} \times \vec{v}' \|}{\| \vec{v} \|^3}$$

Ex. $\sigma(t) = (t, t^2, 0)$

$\vec{v} = \sigma'(t) = (1, 2t, 0)$

$\vec{v}' = \sigma''(t) = (0, 2, 0)$

$$\vec{v} \times \vec{v}' = \begin{vmatrix} i & j & k \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = 2k.$$

the curvature $= \dfrac{2}{(1+4t^2)^{\frac{3}{2}}}$