Table of Contents for the Handbook of Knot Theory

William W. Menasco and Morwen B. Thistlethwaite, Editors

THE ENUMERATION AND CLASSIFICATION OF KNOTS AND LINKS

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Abstract. The theoretical and practical aspects of link classification are described, with special emphasis on the mathematics involved in recent, large-scale link tabulations.

1. Introduction

The first tables of knots were produced in the late 19th century at the very dawn of modern topology and knot theory. Inspired by Lord Kelvin's "Vortex Theory of the Atom" [59], the Scottish physicist P. G. Tait set out to systematically enumerate knots based on their crossing number. Joined later by the English Reverend T. P. Kirkman and America's first knot theorist, C. N. Little, the tabulating trio eventually produced, after untold hours of laborious handwork spread out over a period of about 25 years, a table of prime, alternating knots to 11 crossings and prime non-alternating knots to 10 crossings. A little more than a century later, in July 2003, S. Rankin, O. Flint and J. Schermann tabulated all $6,217,553,258$ prime, alternating knots through 22 crossings in just over a day and a half of computer time! In between lies a story that touches nearly every aspect of the theory of knots—running from the beginning of the subject to the present day, and spanning the entire breadth of the topic in search of the powerful yet practical invariants needed to classify knots.

Much has been written about the history of knot theory and in particular the quest to tabulate knots and links. In this paper we will give only the most basic treatment of the subject, concentrating primarily on the mathematics involved in, and leading up to, the most recent large-scale tabulations. The reader seeking more details is urged to consult the original papers of Tait [55], Kirkman [34], [35], and Little [37], [38], [39], [40]; J. H. Conway [13];

M. B. Thistlethwaite [19], [20], [56]; and recent work by Rankin, Flint and Schermann [47], [48], [49]. These papers represent the core of the tabulating tradition. The early history of knot theory as well as knot tabulation are beautifully described in the historical articles of M. Epple [21], [22], [23], and are highly recommended. A survey of more recent work in knot tabulation may be found in [30].

In Section 2 we review some basic definitions and then outline, in Section 3, the theoretical aspects of knot and link classification. While a complete classification has been achieved, at least in theory, simple means of distinguishing *arbitrary* links with perhaps hundreds or thousands of crossings may never be forthcoming. Section 4 therefore focusses on the practical ingredients of the most recent tabulations—work by Thistlethwaite, J. Hoste and J. Weeks, and now Rankin, Flint and Schermann that have extended the tables into the 20-crossing realm. For lack of space, the details of many important pieces of work, perhaps most noticeably, the groundbreaking and profound contributions of Conway [13], and the promising approach via braids of J. S. Birman and W. W. Menasco [6], are not discussed.

2. Definitions

We assume that the reader has a basic knowledge of knot theory, but we briefly describe the main definitions and important theorems. For a more detailed account see [51].

A *knot* is a smooth, unoriented, embedding of S^1 in S^3, where two such embeddings are considered equivalent if there is a homeomorphism $h : S^3 \to S^3$ which takes one embedded circle to the other. If the homeomorphism h preserves the orientation of S^3, then this is equivalent to saying the embeddings are related by an ambient isotopy. If not, then the two embeddings are related first by a reflection and then by ambient isotopy. No attempt at all will be made at this point to distinguish a knot from its mirror image, or *reflection*, even though such a pair might not be ambient isotopic. A knot which is ambient isotopic to its reflection is called *amphicheiral* or *achiral*. While many knots *obviously* appear to be chiral, it was not until 1914 that topology had developed sufficiently to allow a proof of this! (See M. Dehn [17].)Today, several invariants are known that can easily distinguish many knots from their mirror images. (For example, the Jones polynomial [32].)

A *link* is a disjoint union of knots in S^3, again considered up to homeomorphisms taking one link to another. As before, the homeomorphism may or may not preserve the orientation of S^3. Each knot in the link is called a *component*. A link is called *trivial* if it is the boundary of a disjoint collection of smoothly embedded disks.

Of course, the equivalence relation on knots and links can be made finer by either allowing only ambient isotopy, or by orienting the links and requiring that the orientations be preserved, or both. Additionally, for links, we could order the components and require that the ordering be preserved. None of these refinements will be considered now. Instead, this paper will discuss only the enumeration and classification of unoriented links under the coarsest equivalence defined above. We can then take the point of view of considering these refinements as *symmetries* enjoyed by a particular knot or link. (See [30] for symmetry data of all knots to 16 crossings.)

Just as each integer may be decomposed into a product of primes, so also can each link be expressed in terms of simpler links. The most elementary decomposition occurs when a link is *split*. In this case there exists a smoothly embedded 2-sphere in the link complement which separates some components of the link from the rest. If a link is nonsplit it still might be made up of simpler links via the operation of *connected sum*, which is defined as follows. Given two links L_1 and L_2, in separate copies of S^3, first remove a ball B_i, $i = 1, 2$, from each copy of S^3 which meets each L_i in a diameter of B_i. Now form a new copy of S^3 by gluing together along their boundaries the complementary balls to B_1 and B_2, matching the orientations of the 3-manifolds, and matching the endpoints of what remains of the two links. There are two ways to do this, depending on how the two pairs of endpoints are matched. The newly formed link is the connected sum of L_1 and L_2 and is denoted $L_1 \sharp L_2$. A link is *prime* if it is nonsplit and not the connected sum of nontrivial links.

The operation of connected sum is not well defined. In the case of links having multiple components it clearly matters which components are chosen to be connected together. But additional, more subtle, problems exist too because of our choice of link equivalence. While we make no distinction between a knot K and its reflection $\bar{K}$, it may turn out that $K \sharp J$ and $\bar{K} \sharp J$ are not equivalent. Similarly, if K is an oriented knot and $-K$ is its *reverse* (obtained by reversing the orientation), then even though we consider K and $-K$

the same it may be the case that they are not ambient isotopic by an isotopy respecting orientation. This allows that, *even considered as unoriented knots*, $K \sharp J$ and $(-K) \sharp J$ might not be equivalent, where the connected sum was formed so as to respect the orientations of the summands. (*Nonreversible* knots, or sometimes called *noninvertible* knots, were first shown to exist by H. F. Trotter [60].)

Nevertheless, given a nonsplit, nonprime link we may decompose it as a connected sum and continue to decompose its summands if they remain nonprime until finally the process must end (because of the additivity of genus). Thus every link can be expressed as a connected sum of prime links. In the case of knots, it was proven in 1949 by H. Schubert [53] that the decomposition of a knot into prime summands is unique (up to the ordering of the summands). The analogous result for nonsplit links was shown in 1958 by Y. Hashizume [27]. Because of these results the focus has traditionally been on prime knots and links, since the composite ones can all be built up out of the prime ones.

Every link can be projected into a plane so that the only singularities are a finite number of transverse double points. If at each double point we indicate an over and undercrossing strand in the obvious way then we call the resulting figure a link *diagram*. (See Figure 3.) The minimum number of crossings in any diagram of the same link is called the *crossing number* of the link. Crossings of oriented links can be labeled *right* or *left handed* as follows. Standing on the overcrossing strand of a right handed crossing and facing forward, the undercrossing strand will pass beneath from right to left. Atop a left handed crossing, the strand below runs from left to right.

Representing knots and links by diagrams is undoubtedly the oldest method in use. Many different local diagrammatic changes, some of which preserve the link, and some which do not, are central to the study of knot theory. Chief among these, and shown in Figure 1, are the *Reidemeister moves*, as well as the *flype* shown in Figure 2, which is itself always a combination of Reidemeister moves.

The importance of the Reidemeister moves lies in the fact that two link diagrams represent the same link (here “same” means ambient isotopic) if and only if they are related by a (finite) sequence of such

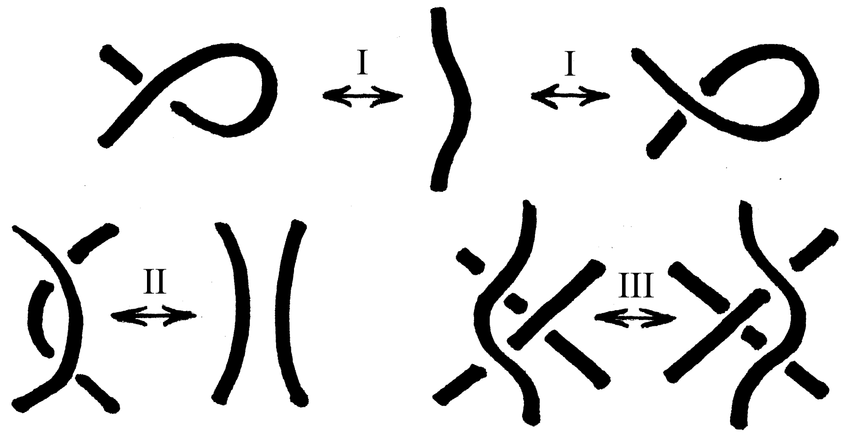


FIGURE 1. Reidemeister Moves

moves[1]. Many important link invariants can be shown to exist by proving that some quantity derived from a link diagram is preserved by Reidemeister moves, and thus actually an invariant of the link type. This diagrammatic, or combinatorial, approach to knot and link theory has been, and continues to be, one of the cornerstones of the subject.

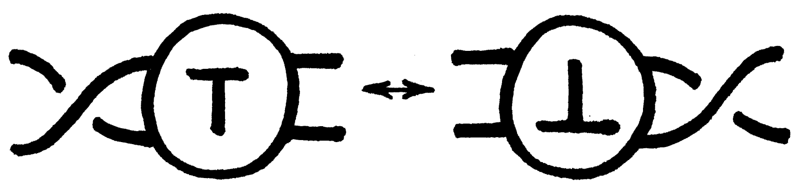


FIGURE 2. A Flype

A crossing in a link diagram is *nugatory* if there is a circle in the projection plane that meets the diagram transversely only at that crossing. A nugatory crossings can clearly be removed by a flype (or perhaps by a single Type I Reidemeister move). A diagram that has no nugatory crossings is called *reduced*.

[1]The Reidemeister moves were known to J. C. Maxwell at least as early as 1868 (See [22].) The proof that they suffice to pass between equivalent diagrams was published by both Reidemeister [50], and J. W. Alexander and G. B. Briggs [2].

3. Classifying Knots and Links

There are infinitely many different knots and links, and to date no practical or simple means to classify them has been found. The emphasis here, of course, must be placed on the words "practical or simple", although even the word "classify" deserves some clarification.

In some sense, the theorem of Reidemeister, and Alexander and Briggs, classifies links according to their diagrams: every link can be represented by a diagram and two such diagrams represent the same link if and only if they are related by Reidemeister moves. But since no *a priori* bound on the number of such moves which might be required to pass between two given diagrams exists, we cannot algorithmically decide, by exploring a finite number of Reidemeister moves, if two diagrams are in fact equivalent.

Alternatively, W. Whitten has shown that prime knots with isomorphic fundamental groups have homeomorphic complements [65]. Coupled with the important theorem of C. McA. Gordon and J. Leucke that knots with homeomorphic complements are equivalent [25], we see that prime knots are classified by their fundamental groups. But as Alexander himself pointed out in 1927 [1], "unfortunately, the problem of determining when two such groups are isomorphic appears to involve most of the difficulties of the knot problem itself."

But again, this is less than satisfactory, as it simply exchanges one difficult problem for another.

To address problems of this sort, we will say that links have been *classified* if we can solve the *recognition problem*. That is, is there an algorithm that can decide, in a finite amount of time, if any given pair of links are equivalent? Notice that given such an algorithm, we could then enumerate all links as follows. Since there are only finitely many link diagrams with a given crossing number, we could systematically list all diagrams arranged by crossing number. As each new diagram is produced, we could compare it to all the diagrams already on the list to see if it represents a new link. If it does, we add it to the list. If not, we discard it. But even this "solution" to the problem leaves us wanting, as ordering the links by crossing number is somewhat artificial, probably having no real bearing on the true topological nature of the links.

In theory, the link recognition problem has been solved. The main work was done by W. Haken, F. Waldhausen, and K. Johannson, with important contributions by G. Hemion, W. Jaco, P. B. Shalen, and S. Matveev. Their algorithm is too complicated to describe completely but we will give a *very brief* description. The interested reader should consult the recent book of S. Matveev [41], where the entire algorithm is described in detail. The algorithm compares the exteriors of the two links which have been additionally marked with the meridians of each component. (The *exterior* of a link is the closure of the complement of a regular neighborhood of the link. A *meridian* of a knot is the boundary of a disk meeting the knot transversely in one point.) Two links are equivalent if and only if there is a homeomorphism between their exteriors taking meridians to meridians. Because of the theorem of Gordon and Leucke, the meridians may be ignored in the case of knots, but in general the Haken algorithm does not rely on this simplification.

In order to determine if the two marked link exteriors are homeomorphic, the fundamental idea is to cut each exterior open along incompressible surfaces and continue to do so with what remains until, eventually, the process must end. By comparing the ending states and their markings, and the regluing instructions needed to return to the link exteriors, it can be decided if the original marked link exteriors are homeomorphic. In order to make the process algorithmic, the link exteriors are first triangulated and the theory of normal surfaces is used to find the incompressible surfaces. Each surface in the link exterior can be assumed to meet each tetrahedron of the triangulation in one of seven basic ways. Assigning seven variables to each tetrahedron to represent the number of local pieces which fit together to form the surface, we arrive at a finite number of matching equations in a finite number of variables. Only a finite number of fundamental solutions to these equations exist and these provide an algorithmic process for constructing the necessary incompressible surfaces in the link exterior. The entire procedure is quite complicated and may never be fully implemented on a computer.

While the Haken algorithm applies to all nonsplit links, the special case of the *unknot recognition* problem has received much attention. This is the problem of deciding if a given knot diagram represents the unknot. The complexity of this problem has been proven to be in class NP by J. Hass and J. C. Lagarias [26]. They also derive from the link recognition algorithm an upper bound on the number of Reidemeister moves m that might be needed to connect a knot diagram D having n crossings to the trivial diagram having zero crossings (assuming D represents the unknot.) They prove that $m \leq 2^{cn}$, where $c = 10^{11}$. It should be noted that Hass and Lagarias made no attempt to find an optimal bound and in fact they believe it is quite likely that the actual upper bound on the required number of Reidemeister moves to connect two equivalent knot diagrams is polynomial in the (larger) number of crossings. Still, more work is clearly needed before the Haken link recognition algorithm becomes practical!

At least one computer program employing normal surface theory has been written. Developed by D. Letscher and B. Burton, and originally called *Normal* but now known as *Regina*, the software can carry out a variety of 3-manifold investigations based on normal surfaces [9]. Using *Regina* it may be possible using Haken's algorithm to identify unknots among diagrams with relatively few crossings.

An algorithmic solution to the unknot recognition problem, which is quite different from the Haken algorithm, has also been found by Birman and M. D. Hirsch [5]. Their approach makes use of braids. Given a knot diagram D with n Seifert circles and c crossings, it is known that D can be redrawn as the closure of a braid β on n strings with at most $c + (n-1)(n-2)$ crossings [61], [66]. They prove that if D is the unknot, then some conjugate of β must be among a certain finite list of n-string braids that depends only on n and c. This list can be generated algorithmically and, making use of the solution to the conjugacy problem in the braid group B_n, each of its elements can be compared to D. A computer program has been written by Birman, M. Rampichini, P. Boldi, and S. Vigna [8] to produce the list of braids. When combined with programs by other authors to solve the conjugacy problem in the braid group (see for example [12]), the entire process should be fully implemented.

Two other computer programs that can be used to attack the unknot recognition program, and which are frequently successful on diagrams with even hundreds of crossings are *SnapPea* by Weeks [62], [64], and *Book Knot Simplifier* by M. Andreeva, I. Dynnikov, S. Koval, K. Polthier and I. Taimanov [4]. Working with a triangulation of the knot exterior, *SnapPea* looks for ways to reduce the number of tetrahedra in the triangulation. *Book Knot Simplifier* makes use of the fact that every link can be embedded in a book with three pages, that is, a union of three half-planes with common boundary. This way of viewing links seems to be particularly adept at computer manipulation. In particular there are a large number of moves, which are easy to locate and apply, that change the 3-page presentation while preserving link type. The program searches for simplifying moves and seems reasonably successful at recognizing unknot diagrams.

Since the Haken link recognition algorithm is impractical, what can we *actually do* when confronted with two link diagrams wanting to know if they are the same link? Fortunately for us, many different knot and link invariants have been developed since the time of Tait, and if the links are different, perhaps some known invariant will distinguish them. Therefore we begin by computing as many invariants as we can, beginning with the easiest to compute and moving to the more difficult. If all known invariants fail to tell the links apart, then perhaps they are the same, and we can launch ourselves into an attempt to relate the two diagrams by Reidemeister moves (or combinations of Reidemeister moves, such as flypes and other moves to be described later). Of course the entire process is *ad hoc*, and may not lead to a definite answer. The harsh reality of knot theory is that we will probably never be able to decide if two arbitrary links are the same or not. Just imagine being given two link diagrams with a few million crossings each!

On the other hand, for certain classes of knots and links, spectacular classification results have been obtained. Wonderful examples include torus links [51], 2-bridge links [51], 3-string braids [7], alternating links and hyperbolic links. The last two classes have proven especially useful to the link tabulator wanting to classify all (prime) links up to a given crossing number. In preparation for the next section, where we describe how the current knot and link tables have actually been constructed, we briefly discuss both alternating and hyperbolic links.

A link is *alternating* if it is represented by a diagram whose crossings alternate, over–under–over–under and so on, as one travels around the components. When Tait first began his investigations he may have thought that all knots were alternating. (Indeed, it is nontrivial to prove that nonalternating knots exist! The first correct proof was given by R. H. Crowell in 1959 [15].) Tait made three conjectures about alternating knots, all of which can be stated for links. The first was that reduced, alternating diagrams have minimal crossing number; the second, that any two reduced alternating diagrams of the same link (here "same" means ambient isotopic) have the same *writhe* (the writhe of a diagram is the number of right handed crossings minus the number of left handed crossings); the third, that two alternating diagrams represent ambient isotopic links if and only if they are related by flypes[2]. (Since flypes preserve writhe, the third conjecture implies the second.) All of these conjectures were eventually proven not long after the discovery of the Jones polynomial, and its generalizations, in 1984. (See, for example, [32], [28], [46], [57], [44], [33], [43].) The proof of the Tait Flyping conjecture by Menasco and Thistlethwaite provides a classification of alternating links. Unlike the Reidemeister moves, the equivalence classes of diagrams related by flypes are all finite. Given two reduced alternating link diagrams, we can algorithmically generate the entire flype equivalence class of one and check to see if it contains the other. Moreover, this task is easily implemented on a computer making the comparison of alternating links reasonably practical. As satisfying as this result is, we must point out that the number of alternating knots and links is small compared to all links. In fact the proportion of links which are alternating tends exponentially to zero with increasing crossing number [54]. Nevertheless, as we will see in the next section, the classification of alternating links provides the important first step in the construction of most link tables.

A truly remarkable situation exists for hyperbolic links. These are links for which the complement admits a complete Riemannian metric of constant curvature -1. The complement of every hyperbolic link can be decomposed in a canonical way into ideal polyhedra. This canonical triangulation depends only on the topology of the link complement. Once the canonical triangulation has been found for two links, the triangulations can be compared combinatorially to decide if the links have homeomorphic complements or not. The canonical triangulation (together with meridian data) provides a complete link invariant! For further details the reader should consult [24], [52], [63].

Unlike the general algorithm of Haken, computing the canonical triangulation of a hyperbolic link complement is much more practical. The program *SnapPea* does just this, as well as compute other invariants of hyperbolic manifolds such as volume, etc. *SnapPea* is so able to handle relatively small knots and links that it was used successfully by Hoste and Weeks in their tabulation with Thistlethwaite of all prime knots through 16 crossings. Unlike

[2]It may be that Tait never actually held this to be true. See [22].

alternating links, the class of hyperbolic links is much "bigger." Of the $1,701,936$ prime knots with 16 or less crossings, all but 32 are hyperbolic. According to Thurston, every knot is either a *torus* knot, a *satellite* knot, or a hyperbolic knot. The torus knots and links are those that can be embedded on a standard, unknotted torus, sitting inside S^3. Torus links are completely classified by how many times they wind in each direction around the torus. Moreover the crossing number of a torus knot or link has been determined by P. B. Kronheimer and T. S. Mrowka [36]. Thus we can tell exactly how many torus links there are of a given crossing number. A satellite link is one that *orbits* a *companion* knot K in the sense that it lies inside a regular neighborhood of the companion. Since every knot is a satellite of the unknot we require the companion to be nontrivial. Note that composite knots are satellites of each of their summands. While few satellites exist at small crossing numbers, their numbers will grow tremendously as the number of crossings increases. If a satellite has *wrapping* number d (that is, the satellite meets every meridional disk in the regular neighborhood of the companion at least d times), then it seems plausible that the crossing number of the satellite is at least cd^2, where c is the crossing number of the companion. Alas, this has not been proven, but it indicates why so few satellites exist to 16 crossings.

How *SnapPea* finds the canonical triangulation of a hyperbolic link complement is described nicely in [64]. However, it is worth pointing out that problems can arise. In order to find the triangulation, certain matching equations must be solved, the solutions of which *SnapPea* only approximates to a high degree of accuracy. Nevertheless, there is no guarantee that the level of accuracy is sufficient to insure that *SnapPea* finds the canonical triangulation, rather than some noncanonical triangulation. Thus, *SnapPea* may falsely declare two hyperbolic links different when in fact they are the same, if it had arrived at the non-canonical triangulation for one or both of the links. But fortunately, it cannot falsely declare two links to be the same. No matter what triangulations are found, canonical or not, if they match for two links, then the link complements are homeomorphic. And if further checking reveals that meridians are taken to meridians, then the links are equivalent. A program called *Snap* has been written by O. Goodman which specifically eliminates the round-off errors introduced by *SnapPea* and therefore cannot make the kind of mistake just described. But unlike *SnapPea*, *Snap* is less practical, taking much longer to find the canonical triangulation for a relatively small knot or link. See [14] for more information on *Snap*.

4. Producing Link Tables

Tabulating knots and links in order to discover the nature of physical matter—Gold is the trefoil! Lead is the figure eight!—was forgotten long ago. But somewhat ironically, tables of knots are now proving useful to scientists once again. For example, knot theory is playing an important role in recombinant DNA research, and researchers in that field need to identify knots and links that occur in their experiments. Important examples in knot theory have also come to light specifically because of the careful and methodical enumeration of knots and links. For example, nontrivial links with trivial Jones polynomial were first found in this way by Thistlethwaite [58], who also turned up the first examples of amphicheiral knots with odd crossing number [30]. No doubt other interesting and important examples will surface as the tables are extended even further. Having billions of links in the tables, as opposed to only a few hundred, provides a much richer and realistic data set for experimenting and testing conjectures. Thus tables are an important part of our field and are likely to expand further as new algorithms and invariants are discovered and computers grow ever faster.

We outline here the basic plan that has been used to create the latest (and largest) tables of prime knots and links. The idea, already mentioned earlier, is simple: systematically list all diagrams to a given crossing number and then group them together according to link type. Because practical link recognition can only be carried out in an *ad hoc* way (except for alternating links) the second half of the program is clearly the more vexing part. But given the huge number of diagrams possible, even the first part must be undertaken with some care.

Once a table of prime knots and links has been found, these can then be connect-summed together in all possible ways to obtain the composite links. Thus efforts have concentrated primarily on tabulating prime knots and links. It is worth pointing out though that one of the more obvious "theorems" of knot theory, that crossing number is additive under connected sum, has yet to be proven (or, incredibly, could it be false?) So while our table of prime knots and links might be arranged by crossing number, it is conceivable that using it to produce the composite links might not create them in order with respect to crossing number.

Our overall plan is thus the following. First enumerate all prime alternating link diagrams to a given crossing number and group these together by flype equivalence class so as to create a table of prime alternating links. Three theorems are of particular importance here. The first Tait Conjecture assures us that a reduced alternating diagram is minimal in crossing number. Hence we can be sure that the crossing numbers of the knots and links represented by the diagrams under consideration are exactly what they appear to be. The second theorem is an important result of Menasco that states that a reduced alternating diagram represents a prime link if and only if it is prime as a diagram [42]. That is, if no circle in the projection plane meets the diagram transversely in two points with crossings on either side, then in fact the link is prime. This result is an incredible boon to the tabulator, allowing easy recognition of composite alternating links. Finally, the already mentioned Tait Flyping Conjecture obviously plays a crucial role.

After the alternating links have been tabulated, we consider a diagram representing each alternating link, and change its crossings in all possible ways in order to create nonalternating diagrams. Most of these will reduce to fewer crossings, and some link diagrams may even represent split links or composite links. So clever methods will be needed to weed out the multitude of uninteresting and unwanted diagrams. Finally, *ad hoc* methods must be employed to either distinguish all that remains (by computing various invariants), or recognize repeats (by finding sequences of Reidemeister moves or, in the case of hyperbolic links, comparing canonical triangulations).

The entire process begins with a scheme to encode diagrams.

4.1. **Encoding Link Diagrams.** There are a variety of ways to encode knot and link diagrams. One scheme, which was introduced by Tait (and was similar to ideas of C. F. Gauss and J. B. Listing), and further refined by C. H. Dowker and Thistlethwaite, has proven to be quite useful. Referred to here as the *DT sequence* of a link diagram (after Dowker and Thistlethwaite), it has been used successfully by a number of people in compiling modern tables [19], [20], [56], [3], [30]. The primary advantage of the DT sequence seems to be its brevity.

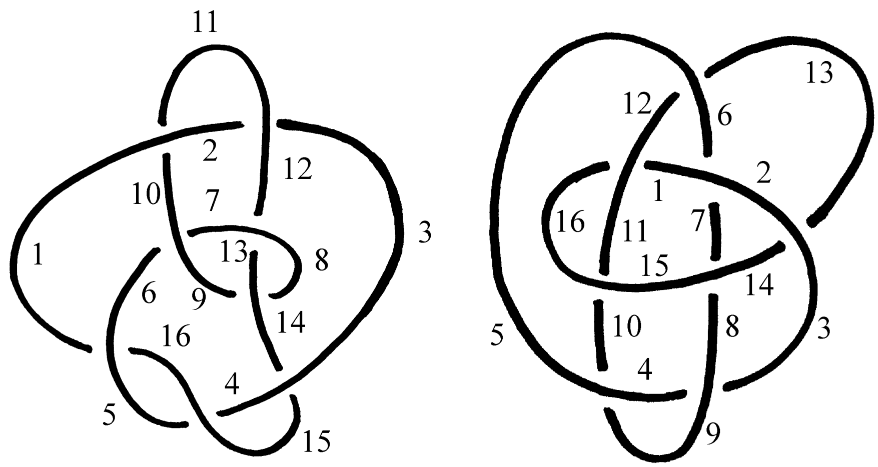
With over 6 billion knots and links now in the tables, using as little computer memory as possible has obvious value. On the other hand, DT sequences are not easily transformed *directly* under the types of operations that need to be applied to diagrams in the course of a tabulation. Instead, it is usually necessary to derive additional, attendant information about the diagram, such as the signs of the crossings, or how the faces of the diagram (the complementary regions) meet one another, and so on, before operating on the diagram. Using this derived information assists in encoding new diagrams obtained from old ones via changes such as flypes, or other Reidemeister moves. Not only storage size, but also computing speed are of great importance in any computer assisted tabulation. Therefore it usually makes sense to sacrifice brevity in favor of a more detailed and redundant encoding scheme if it can increase the speed at which diagrams can be manipulated. Indeed, the recent tabulation of alternating knots through 22 crossings by Rankin, Flint and Schermann uses an encoding scheme that is far more complicated than the DT sequence, takes more storage space per link, yet seems to be more finely tuned to the task of enumeration. Because of its historical importance, as well as continuing utility in modern tabulations and knot manipulation software, we will describe the DT sequence in some detail. Before doing so however, we briefly mention other encoding schemes of importance.

A quite different method of notation was introduced by Conway [13], building on ideas used by Kirkman. Conway's scheme is quite efficient for links of low crossing number and in fact reflects deep structural properties of links. But it draws on a large set of symbols arranged according to a rather large set of rules, both of which grow with crossing number and for this reason does not lend itself well to computer programming.

However, for small knots and links Conway's system is so efficient that it allowed him to tabulate (by hand) all prime knots to 11 crossings and all prime links to 10 crossings in a few hours! Conway found 11 omissions and one duplication in Little's table of 11 crossing alternating knots. In the late 1970's A. Caudron [11] used an alternative version of Conway's notation to retabulate all prime knots to 11 crossings, discovering, in the process, four omissions in Conway's list of 11 crossing nonalternating knots. Continuing with the methodology of Kirkman and Conway, S. Jablan [31] completed the tabulation of 12 crossing alternating links in 1997.

Every link may be represented as a closed braid and so braid notation is an obvious choice for encoding diagrams. However, since transforming an arbitrary diagram into a closed braid usually involves an increase in the number of crossings, braids are perhaps not the best choice for a table organized by crossing number. But perhaps organizing a table instead with respect to one of the indices appropriate for braids, such as number of strings, would be better. At any rate, no major tabulation has yet to be undertaken based on braids. Perhaps the work of Birman and Menasco [6], coupled with the unknot recognition algorithm of Birman and Hirsch [5], or similarly inspired algorithms, will one day lead to a major tabulation using braids. Instead, some effort has been made to systematically find braid representatives for already tabulated knots and links.

To encode a diagram with a DT sequence, first consider an arbitrary knot diagram with n crossings, and therefore $2n$ edges. (The edges are the components of the associated projection minus the double points.) Place a basepoint on one of the edges and also choose an orientation of the knot. We may now label the crossings with consecutive integers $1, 2, 3, \ldots, 2n$ as we travel around the projection starting from the basepoint. Each crossing receives two labels and it is a consequence of the Jordan Curve Theorem that the labels at each crossing have opposite parity. (A slight variation is to label the edges rather than the crossings. In this case the edge labels are paired, one with another, by seeing which two labels lead into each crossing.) The pairing of labels at each crossing gives a permutation, σ of the set $\{1, 2, 3, \ldots, 2n\}$. The sequence of even numbers, $S = \{\sigma(1), \sigma(3), \ldots, \sigma(2n-1)\}$ is sufficient to denote σ. The final step in producing the DT sequence is to consider how the diagram differs from an alternating diagram. If the diagram is alternating then S is used to denote the diagram. If not, then some set of crossings of the diagram may be changed to produce an alternating diagram. The labels of S corresponding to these crossings are then negated to finally give a signed sequence of the even integers from 2 to $2n$. Since there are two possible alternating diagrams possible (each the reflection of the other), we see that S is only defined up to negation of all its entries. Since our aim is only to tabulate knots up to reflection, this presents no difficulty. It makes sense to either choose S to have the fewest number of minus signs, or to begin with a positive integer. The process is illustrated in Figure 3.



1	3	5	7	9	11	13	15
10	14	16	12	6	2	8	4

eghfcadb

1	3	5	7	9	11	13	15
6	-8	12	-14	-4	16	-2	10

cDfGBhAe

FIGURE 3. DT Sequences for Knot Diagrams

Since there are $4n$ possible choices of basepoint and direction, it is quite possible that all $4n$ DT sequences will be different. Choosing the lexicographically smallest one provides the *minimal* DT sequence for the diagram. A computer routine that finds the minimal DT sequence equivalent to a given one is easy to write.

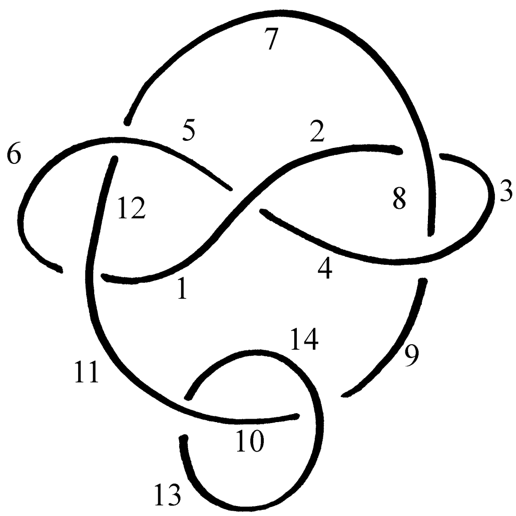
Two problems arise regarding DT sequences. The first is that for an arbitrary diagram it may not be possible to recover the diagram from the sequence. However, it is proven in [19] that if the diagram represents a prime knot, and has no nugatory crossings, then it is determined by its sequence, although only up to reflection and isotopy in S^2. Again this presents no difficulty since these distinctions will not be made among knots in the table. The second difficulty, and one that presents a particularly annoying computing problem, is that most signed sequences of the numbers $\{2, 4, \ldots, 2n\}$ are not *realizable*, that is, do not correspond to any knot. A moment's thought reveals that what DT sequences really record are (certain) 4-valent graphs and of course, not all graphs are planar. While it is not difficult to decide if a "DT sequence" really is one, it is time consuming to do so. However, using the basic tabulation scheme of J. A. Calvo [10] described later, we'll see that it will never be necessary to test arbitrary DT sequences to decide if they in fact represent knots.

Notice that the DT sequence of a knot with n crossings may be stored on a computer by using $2n$ bytes, since two bytes are usually used to store an integer. However, we can easily halve this to n bytes by using characters rather than integers, since one byte is typically used to store a character. A particularly nice scheme is to use "a" and "A" for 2 and -2, "b" and "B" for 4 and -4, etc. This allows for up to 26 crossings, but could obviously be extended considerably by using the full ASCII character set of 256 characters (and an agreed upon pairing of those characters.) Thus DT sequences provide a fairly compact notation for diagrams.

The scheme can be extended to links by choosing an orientation and basepoint for each component and then numbering the crossings (or edges) consecutively, beginning at each basepoint in turn. In general this may not produce the desired even-odd pairing at each crossing, but it is not hard to see that some choice of basepoints will. Restricting ourselves to these labellings we once again get a sequence of even integers that may be further refined by introducing minus signs to designate where the diagram differs from alternation. However, one more piece of information is needed in the DT sequence, namely which labels lie on the same component. This can be denoted by inserting vertical bars (or some other character) into the sequence to separate components. An example is given in Figure 4.

Again, as with knots, many signed sequences with bars do not represent actual links. Moreover, in general, it may not be possible to recover the diagram from the sequence without some serious ambiguity. But it is proven in [18] that if a DT sequence encodes a reduced diagram of a prime, nonsplit link, then it determines the diagram up to reflection and isotopy in S^2. Since we only wish to tabulate unoriented, prime, non-split links, up to reflection, we once again can avoid any possible ambiguity arising from the encoding scheme.

For the class of nonsplit links, the number of possible choices of basepoints and orientations can be reduced considerably. After the first basepoint and orientation are chosen there are a number of possible ways to then determine uniquely all the remaining basepoints and orientations. One example is as follows. After choosing the first basepoint and orientation of that component, change crossings if necessary to make the diagram begin with an overcrossing and also alternate. Number the crossings of the first component as usual.



1 3 5 7 9 11 13
4 8 12 | 2 -14 6 | -10
bdf | aGc | E

FIGURE 4. A DT Sequence for a Link Diagram

Because the link is non-split, the first component must cross other components and therefore some crossings of the first component have so far received only one label. Among these, pick the one, c, with the smallest label. Orient the other component at crossing c so that c is right handed, and place the second basepoint on that component so that c receives the smallest possible label such that the two labels at c have opposite parities. If any components remain we treat them in exactly the same way, continuing until the entire diagram has been labeled. This gives the sequence of even integers, and changing crossings to return to the original diagram indicates where minus signs must be inserted. Finally, bars are used to separate the components. The example in Figure 4 obeys this scheme. By restricting ourselves to this algorithm (or a similar one) we once again have $4n$ possible ways to encode a link diagram. To choose the lexicographically smallest DT sequence for the diagram, some consideration must be made of the bars. A nice choice is to regard the bar as zero (in fact, this is a nice way to store a DT sequence in the computer), and then use ordinary lexicographic order.

4.2. **Generating All Alternating Diagrams.** Each (unsigned) DT sequence corresponds to an alternating link diagram (actually, to a projection). Thus to generate all possible alternating diagrams with n crossings we simply need to consider all possible DT sequences of length n. This is exactly the original approach taken by Tait as well as the computer equipped tabulators of the late 20-th century [30], [3], [18], [16]. But this approach has several major drawbacks that make it more and more impractical as n grows to 16, 17, and beyond. The main problem is that most DT sequences do not encode prime, nonsplit links. Thus considerable time is spent testing each DT sequence to see if it is valid, with the greatest amount of time spent on deciding if the sequence is realizable. Even though some clever tricks can be introduced to avoid testing all possible DT sequences, huge amounts of time are still wasted considering useless DT sequences.

Instead a significant savings can be achieved by inductively generating the n-crossing alternating diagrams from the k-crossing diagrams, where $k < n$. The basic idea is due to Calvo and K. Millett [10], and has been successfully used by the author to tabulate all alternating knots to 18 crossings, and by Thistlethwaite to tabulate all alternating knots and links to 19 crossings. A more refined version, which we will briefly describe later, has been used by Rankin, Flint and Schermann to tabulate all alternating knots to 22 crossings.

Suppose that D is a reduced, prime, alternating link diagram of n crossings. By *smoothing*, or *nullifying*, a crossing we may transform D to a diagram, D', of one less crossing. The two possible ways to smooth a crossing are illustrated in Figure 5.

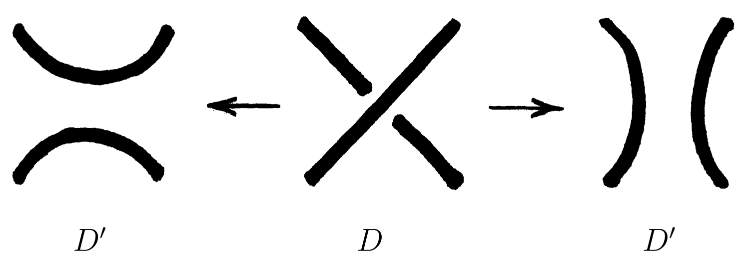


FIGURE 5. Smoothing a Crossing

Depending on whether the crossing was a *pure* crossing (between two strands of the same component) or a *mixed* crossing (between two strands of different components), and which of the two possible smoothings is performed, the number of components of D and D' may be equal or differ by one. But regardless, D' represents an alternating link of at least one less crossing and therefore must already be in our census, provided it is prime. While D was assumed to be reduced, this need not be the case for D'. After the smoothing, nugatory crossings may be present that can then be eliminated. Figure 6 illustrates a possible scenario.

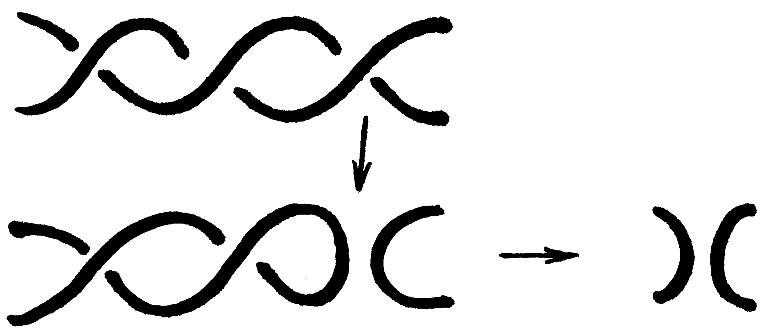


FIGURE 6. Smoothing and Reducing

The important result of Calvo is that there is always at least one crossing in D that can be smoothed so that the resulting diagram D' will reduce, after eliminating nugatory crossings, to a reduced, prime, alternating link diagram. Thus we may inductively build the collection of all prime, reduced, alternating link diagrams of n crossings by starting from the collection of all prime, reduced, alternating link diagrams with fewer crossings and splicing in twisted bands as in Figure 6 in all possible ways. Note that with this approach there is no reason to tabulate knots separately from links.

Using Calvo's algortihm, no testing of DT sequences for realizability, primality, or nugatory crossings is ever required! On the other hand, there is tremendous redundancy among the diagrams that are produced, especially if no attempt is made to account for the flype-structure of a diagram. However, Calvo also describes the general flype structure of a reduced prime alternating diagram. Each crossing c that can be involved in a nontrivial flype (and not all crossings can be) generates a unique *flype cycle* as shown in Figure 7. Each flype tangle on the circuit (represented by a disk in the Figure) is minimal in the sense that flipping it over cannot be achieved as a sequence of "smaller" flypes. A diagram is in *flype minimal* position if all crossings that generate the same flype cycle are grouped together in a single twisted band between two flype minimal tangles. (The diagram in Figure 7 is not flype minimal.) For a nonsplit, prime, alternating link L we may form a graph using flype minimal diagrams of L as vertices and connecting two vertices if they are related by a minimal flype (not the composition of smaller flypes). Calvo shows that this graph is always an f-dimensional torus lattice, where f is the number of flype cycles in the link. Understanding the flype structure of a prime, alternating diagram, can greatly increase the efficiency of the overall program to inductively create all prime, alternating links.

Calvo's program has been carried out by Thistlethwaite for all prime, alternating links to 19 crossings, and independently by Hoste (for knots only), to 18 crossings. Their results agree with each other, and with the independent tabulation of Rankin, Flint and Schermann of prime alternating knots to 22 crossings.

In the remainder of this section we will briefly describe the Rankin, Flint and Schermann tabulation, which is essentially a refinement of Calvo's approach with special effort having been taken to avoid redundant work thereby increasing efficiency. Their approach is extremely technical and the interested reader should consult their papers for details [47], [48], [49].

Rankin, Flint and Schermann consider four diagrammatic operations which they call D, ROTS, T, and OTS and which are pictured in Figure 8. These operations are applied to prime, alternating knot diagrams. Since the input diagrams are alternating, we have drawn

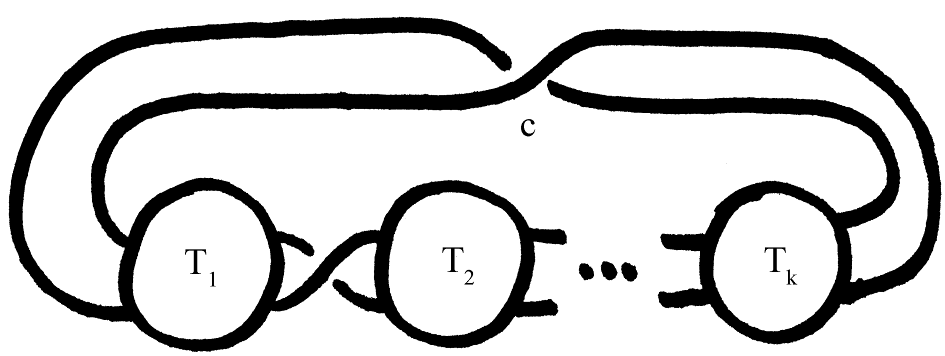


FIGURE 7. Crossing c and its Flype Cycle

only projections in the figure, not bothering to indicate the possible arrangements of crossings. However, given a choice of crossings in the input, the output must have its crossings chosen so as to remain alternating. The input diagrams are also unoriented, but in the case of the D operator, an orientation must be introduced in order to correctly apply D. Similarly, T is only applied if the orientation of the input matches that shown in the figure.

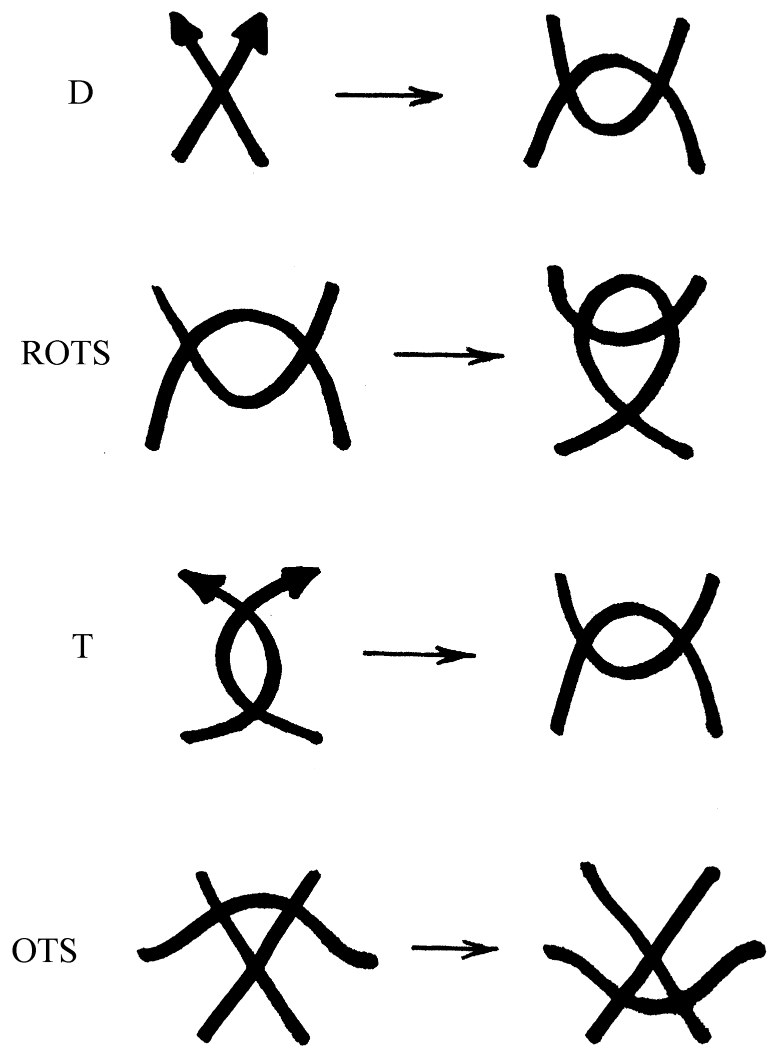
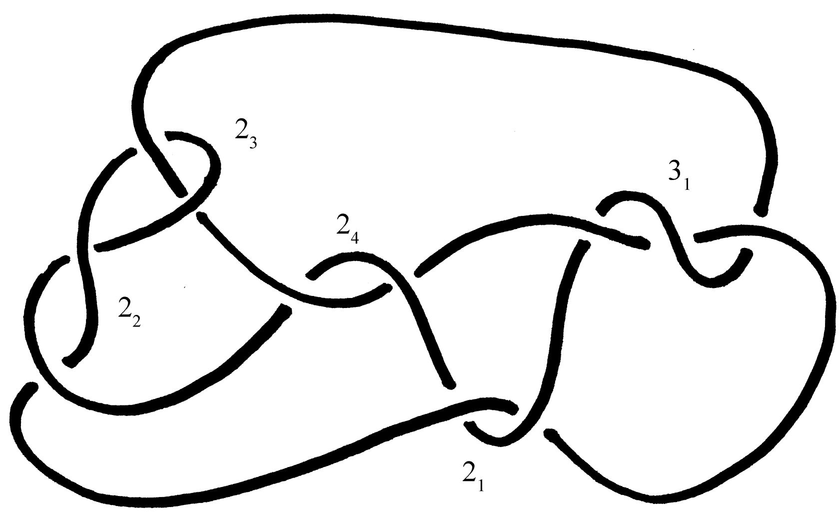


FIGURE 8. The D, ROTS, T, and OTS operators.

The basic idea, as with Calvo, is to inductively build up the $n+1$ crossing knots from the n crossing knots. Given all prime, alternating n crossing knots, D and ROTS are first applied to build a collection of $n+1$ crossing, prime, alternating knots. After this is done, T and OTS are repeatedly applied to the collection until no new knots appear.

In order to make the process more efficient by avoiding the creation of redundant diagrams, Rankin, Flint and Schermann make a careful analysis of the flype cycle structure of a knot and introduce a refined knot encoding scheme that contains this information. Their method is reminiscent of Conway's idea of inserting tangles at 4-valent vertices of graphs. Consider the knot shown in Figure 9. In several locations there are sets of crossings obtained by twisting a pair of parallel strands. Rankin, Flint and Schermann call these *groups* of crossings, and a k-*group* is a maximal such set of k crossings. For each k, the k-groups are labeled $k_1, k_2, \ldots k_{i_k}$ and a type of Gauss code is then recorded as one traverses the knot. Starting from some basepoint, and traveling in some direction, the labels of the groups are recorded as they are passed. Furthermore, each group is either *positive* or *negative* depending on whether the two strands in the group are oriented parallel or anti-parallel. Minus signs are then inserted into the Gauss code accordingly. The *group code* for the diagram in Figure 9, as it is called by Rankin, Flint and Schermann, is given in the Figure.



$$-2_1, -2_2, 2_3, -2_2, 2_4, -2_1, 3_1, 2_3, 2_4, 3_1$$

FIGURE 9. A diagram with k-group encoding.

As can be seen in Figure 7, all crossings that share the same flype cycle can be brought together by flyping into a (maximal) k-group. Assuming that this has been done for every flype cycle, Rankin, Flint and Schermann then show that it suffices to operate only on these flype minimal diagrams. They furthermore show that the four operations need only be applied as follows. First D is applied to any one crossing in each negative k-group, $k \geq 1$, and in each positive 2-group in each n crossing knot. After all possible applications of D have been made, the ROTS operator is then applied to each negative 2 or 3 group in every n crossing diagram. Rankin, Flint and Schermann remark that at this point they have usually produced about 98% of the $n+1$ crossing diagrams. For example, of the 40,619,385 prime, alternating 19 crossing knots, 39,722,121 were found after only applying D and ROTS.

Next T is applied to each positive 2-group in all of the $n+1$ knots that have been created so far, and this is continued until no new knots are added. The OTS operator is then applied until it also produces no new knots. The operations of T and OTS are then repeated alternately until no new knots result. At his point the construction of all $n+1$ prime, alternating knots is complete.

An important feature of their work, which is too complicated to explain here, is that they employ a more sophisticated data structure than simply the group code explained earlier. Instead they record the flype cycle information for each diagram in what they call the *master group code* and among all such encodings choose one, called the *master array* to represent the knot. Two diagrams are flype equivalent if and only if they have identical master arrays.

4.3. **Generating the Nonalternating Diagrams.** If a prime, alternating diagram D has has n crossings, then we may produce 2^n nonalternating diagrams by switching crossings in all possible ways. Of course, half of these are unnecessary since we will consider any link and its mirror image as the same. Most of these diagrams will reduce to fewer crossings, and possibly even represent split or composite links. One way to avoid generating unwanted diagrams is to first group all the crossings of D into subsets where within each subset all the crossings must maintain the same state relative to each other lest an immediate reduction to fewer crossings is possible. For example, the set of crossings associated to a twisted band, or in the language of Rankin, Flint and Schermann, a k-group, is such a set. If all the crossings of D can be partitioned into j subsets of this kind then only 2^{j-1} vs 2^{n-1} nonalternating diagrams need be considered. Assuming that the nonalternating links of up to $n-1$ crossings have already been tabulated, we may first eliminate any diagram that reduces to fewer crossings. Note that while many diagrams will reduce to fewer crossings, relatively few will be unknots. Thus attempting to apply one of the unknot recognition algorithms discussed in the last section would probably be unwarranted at this point. On the other hand, perhaps employing the ideas of I. Dynnikov and the 3-page book simplification moves could prove useful. At present, the only large-scale tabulations of nonalternating links have been carried out by Thistlethwaite (knots and links), and Hoste and Weeks (knots only), and in both cases a variety of Reidemeister moves were employed in an effort to eliminate diagrams that reduce to fewer crossings. In the author's case, the equivalence class generated by flypes and 2-passes was generated for each diagram and each of these diagrams was searched for (i,j)-pass moves which would reduce crossing number. An (i,j)-pass move is illustrated in Figure 10. This move "picks up" a bridge with i overcrossings and "lays it down" in a new location with j overcrossings. A 2-pass is a $(2,2)$-pass. Thistlethwaite uses these moves plus many other esoteric moves (for example, one derived from the "Perko pair" equivalence [45]) to search for reductions. A more comprehensive description of Thistlethwaite's moves may be found in [30].

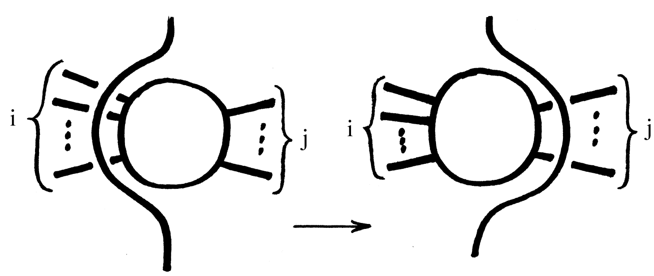


FIGURE 10. The (i,j)-pass move.

It is important to note that while lots of theoretical algorithms might be brought to bear on this stage of the problem the sheer number of link diagrams under consideration requires great economy. Clever programming and *ad hoc* tricks might bring about greater gains than sophisticated algorithms derived from the most powerful theorems in topology. A certain side of tabulation remains an art form.

Experience shows that through about 17 crossings, flypes and pass moves appear to suffice to reduce the list of nonalternating links down to about 110-120% of its eventual size. At this point the computation of various link invariants can be undertaken. Through 17 crossings, at least for knots, the computation of hyperbolic structure with *SnapPea* worked well. Very few knots were identified as nonhyperbolic, and very few hyperbolic knots had volumes so suspiciously close that further inspection (with other invariants) was warranted. The situation with links is not quite as favorable—many more pairs are too close to call with *SnapPea* and many more nonhyperbolic links exist. The winnowing out of duplicates from Thistlethwaite's nonalternating lists of links through 19 crossings still awaits completion.

	Number of Components								
n	1	2	3	4	5	6	7	8	9
0	1								
1									
2		1							
3	1								
4	1	1							
5	2	1							
6	3	3	2						
7	7	6	1						
8	18	14	6	1					
9	41	42	12	1					
10	123	121	43	9	1				
11	367	384	146	17	1				
12	1288	1408	500	100	11	1			
13	4878	5100	2074	341	23	1			
14	19536	21854	8206	1556	181	13	1		
15	85263	92234	37222	7193	653	29	1		
16	379799	427079	172678	33216	3885	301	16	1	
17	1769979	2005800	829904	173549	19122	1129	36	1	
18	8400285	9716848	4194015	876173	105539	8428	471	19	1
19	40619385	48184018	21207695	4749914	599433	43513	1813	43	1
20	199631939								
21	990623857								
22	4976016485								

TABLE 1. Number of prime unoriented alternating links per crossing number n and number of components.

5. Conclusion

Presently over 6 billion knots and links have been tabulated, some with crossing number as high as 22. The numbers of prime, unoriented, alternating links per crossing number and component number are given in Table 1. The corresponding numbers for nonalternating links are given in Table 2. As this article is being written, Rankin, Flint and Schermann are preparing to tabulate the 23-crossing prime alternating knots, and furthermore to turn their attention to alternating links. Thistlethwaite's alternating link tables to 19 crossings have yet to be confirmed, and his lists of nonalternating links still await the final removal of duplicates.

Tables of prime knots through 16 crossings are widely available in the software package *Knotscape* [29] written by Hoste and Thistlethwaite. This software not only contains the tables, but will compute various knot invariants and locate knots in the tables. A major revision of *Knotscape* that will also handle links and include knot and link tables to 17 crossings is currently underway. As of this writing, the tables of Rankin, Flint and Schermann are not yet publicly available.

How far will this current burst of tabulation take us? To 25 or 30 crossings? Will a mega-tabulation distributed over thousands of machines via the internet be organized? Will a knot with trivial Jones polynomial, or some other surprising example be found this way? Perhaps the greatest gains in tabulation will result from improvements in computers, but no doubt theoretical advances will be made as well, allowing more efficient algorithms, and providing better invariants. This is an exciting time in knot theory!

Acknowledgement Special thanks are due to Joel Hass, Sergei Matveev, Morwen Thistlethwaite, and Jeff Weeks for their helpful comments.

	Number of Components			
n	1	2	3	4
0				
1				
2				
3				
4				
5				
6			1	
7		2		
8	3	2	4	2
9	8	18	9	0
10	42			
11	185			
12	888			
13	5110			
14	27436			
15	168030			
16	1008906			

TABLE 2. Number of prime, unoriented, nonalternating links per crossing number n and number of components.

REFERENCES

[1] J. W. Alexander, *Topological invariants of knots and links*, Trans. Am. Math. Soc. **30** (1928), no. 2, 275–306.

[2] J. W. Alexander and G. B. Briggs, *On types of knotted curves*, Ann. Math. **28** (1927), 562–586.

[3] B. Arnold, M. Au, C. Candy, K. Erdener, J. Fan, R. Flynn, J. Hoste, R.J. Muir, and D. Wu, *Tabulating alternating knots through 14 crossings*, J. Knot Theory Ramifications **3** (4) (1994), 433–437.

[4] M. Andreeva, I. Dynnikov, S. Koval, K. Polthier and I. Taimanov, *Book Knot Simplifier*, computer software, http://www-sfb288.math.tu-berlin.de/vgp/javaview/services/knots/.

[5] J. S. Birman and M. D. Hirsch, *A new algorithm for recognizing the unknot*, Geometry and Topology **2** (9) (1998), 175–220.

[6] J. S. Birman and W. W. Menasco, *A calculus on links in the* 3*-sphere*, Knots 90 (Osaka, 1990), 625–631, de Gruyter, Berlin, 1992.

[7] J. S. Birman and W. W. Menasco, *Studying links via closed braids. III. Classifying links which are closed* 3*-braids*, Pacific J. Math. **161** no. 1 (1993), 25–113.

[8] J. S. Briman, M. Rampichini, P. Boldi and S. Vigna, *Towards an implementation of the B–H algorithm for recognizing the unknot*, J. Knot Theory Ramifications **11** (4) (2002), 601–645.
[9] B. Burton and D. Letscher, *Regina*, computer software, http://regina.sourceforge.net.
[10] J. A. Calvo, *Knot enumeration through flypes and twisted splices*, J. Knot Theory and Its Ramifications **6** (6) (1997), 785–797.
[11] A. Caudron, *Classification des noeuds et des enlacements*, Prepublication Math. d'Orsay, Orsay, France: Universite Paris-Sud (1981).
[12] J. C. Cha, K. H. Ko, S. J. Lee, J. W. Han and J. H. Cheon, *An efficient implementation of braid groups*, *Asiacrypt*, Lecture Notes in Computer Science 2248 (2001), 157–174.
[13] J. H. Conway, *An enumeration of knots and links and some of their algebraic properties*, Computational problems in abstract algebra (Proc. Conf., Oxford, 1967), 329–358 Pergamon, Oxford.
[14] D. Coulson, O. Goodman, C. Hodgson and W. Neumann, *Computing arithmetic invariants of 3-manifolds*, Experiment. Math. **9** (1) (2000), 127–152.
[15] R. H. Crowell, *Nonalternating links*, Ill. J. Math. **3** (1959), 101–120.
[16] O. T. Dasbach and S. Hougardy, *Does the Jones polynomial detect unknottedness?*, Exp. Math. **6** no. 1 (1997), 51–56.
[17] M. Dehn, *Die beiden Kleeblattschlingen*, Math. Ann, 75 (1914), 402–413.
[18] H. Doll and J. Hoste, *A tabulation of oriented links*, Math. Computat. **57** (196) (1991), 747–761.
[19] C. H. Dowker and M. B. Thistlethwaite, *On the classification of knots*, C. R. Math. Rep. Acad. Sci. Canada **4** (1982), no. 2, 129–131.
[20] C. H. Dowker and M. B. Thistlethwaite, *Classification of knot projections*, Topol. Appl. **16** (1983), 19–31.
[21] M. Epple, *Branch points of algebraic functions and the beginnings of modern knot theory*, Historia Math. 22 (1995), 371–401.
[22] M. Epple, *Topology, matter and space. I. Topological notions in 19th-century natural philosophy*, Arch. Hist. Exact Sci. **52** (4) (1998) 297–392.
[23] M. Epple, *Die Entstehung der Knotentheorie: Kontexte und Konstruktionen einer modernen mathematischen Theorie*. Wiesbaden: Vieweg, 1999.
[24] D. B. A. Epstein and R. C. Penner, *Euclidean decompositions of noncompact hyperbolic manifolds*, J. DIff. Geom. **27** (1988), 67–80.
[25] C. McA. Gordon and J. Luecke, *Knots are determined by their complements*, J. Am. Math. Soc. **2** (1989), 371–415.
[26] J. Hass and J. C. Lagarias, *The number of Reidemeister moves needed for unknotting*, J. Amer. Math. Soc. **14** (2001), no. 2, 399–428.
[27] Y. Hashizume, *On the uniqueness of the decomposition of a link*, Osaka Math. J. **10** (1958) 283–300.

[28] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. **12** (1985), 239–249.

[29] J. Hoste and M. B. Thistlethwaite, *Knotscape*, computer software, http://www.math.utk.edu/˜morwen.

[30] J. Hoste, M. B. Thistlethwaite and J. Weeks, *The first 1,701,936 knots*, Math. Intelligencer **20** (4) (1998), 33–48.

[31] S. Jablan, *Geometry of links*, XII Yugoslav Geometric Seminar (Novi Sad, 1998). Novi Sad J. Math. **29** no. 3 (1999), 121–139.

[32] V. F. .R. Jones, *A polynomial invariant for knots via Von Neumann algebras*, Bull. Amer. Math. Soc. **12** (1985), 103–111.

[33] L. H. Kauffman, *State models and the Jones polynomial*, Topology **26** (1987), 395–407.

[34] T. P Kirkman, *The enumeration, description and construction of knots of fewer than ten crossings*, Trans. Roy. Soc. Edinburgh **32** (1885), 281–309.

[35] T. P Kirkman, *The 364 unifilar knots of ten crossings enumerated and defined*, Trans. Roy. Soc. Edinburgh **32** (1885), 483–506.

[36] P. B. Kronheimer and T. S. Mrowka, *Gauge theory for embedded surfaces. I.*, Topology **32** (1993), no. 4, 773–826.

[37] C. N. Little, *On knots, with a census of order ten*, Trans. Connecticut Acad. Sci. **18** (1885), 374–378.

[38] C. N. Little, *Non alternate ± knots of orders eight and nine*, Trans. Roy. Soc. Edinburgh **35** (1889), 663–664.

[39] C. N. Little, *Alternate ± knots of order 11*, Trans. Roy. Soc. Edinburgh **36** (1890), 253–255.

[40] C. N. Little, *Non-alternate ± knots*, Trans. Roy. Soc. Edinburgh **39** (1900), 771–778.

[41] S. Matveev, *Algorithmic Topology and Classification of 3-manifolds. Algorithms and Computation in Mathematics*, **9**, (2003), Springer.

[42] W. W. Menasco, *Closed incompressible surfaces in alternating knot and link complements*, Topology **23** (1) (1984), 37–44.

[43] W. W. Menasco and M. B Thistlethwaite, *The classification of alternating links*, Ann. Math. **138** (1993), 113–171.

[44] K. Murasugi, *The Jones polynomial and classical conjectures in knot theory*, Topology **26** (1987), 187–194.

[45] K. Perko, *On the classification of knots*, Proc. Am. Math. Soc. **45** (1974), 262–266.

[46] J. H. Przytycki and P. Traczyk, *Invariants of links of Conway type*, Kobe J. Math. **4** (1987), 115–139.

[47] S. Rankin, O. Flint and J. Schermann, *Enumerating the prime alternating knot, part I*, J. Knot Theory Ramificat. **13** (1) (2004), 57–100.

[48] S. Rankin, O. Flint and J. Schermann, *Enumerating the prime alternating knot, part II*, J. Knot Theory Ramificat. **13** (1) (2004), 101–150.

[49] S. Rankin and O. Smith, *Enumerating the prime alternating links*, J. Knot Theory Ramificat. **13** (1) (2004), 151–173.

[50] K. Reidemeister, *Elementare Begründung der Knotentheorie*, Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität **5** (1926), 24–32.

[51] D. Rolfsen, *Knots and Links*, Mathematics Lecture Series 7, Publish or Perish, Houston, TX (1990).

[52] M. Sakuma and J. Weeks, *The generalized tilt formula*, Geometriae Dedicata **55** (1995), 115–123.

[53] H. Schubert, *Die eindeutige Zerlegbarkeit eines Knotens in Primknoten*, S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1949, (1949). no. 3, 57–104.

[54] C. Sundberg and M. .B. Thistlethwaite, *The rate of growth of the number of prime alternating links and tangles*, Pacific J. Math. **182** (2) (1998), 329–358.

[55] P. G. Tait, *On knots I, II, III*, Scientific Papers, Vol. I, Cambridge: Cambridge University Press, 273–347.

[56] M. B. Thistlethwaite, *Knot tabulations and related topics*, Aspects of Topology in Memory of Hugh Dowker 1912–1982 (I.M. James and E.H. Kronheimer, eds.) Cambridge, England: Cambridge Press, London Math., Soc. Lecture Note Series 93 (1985),1–76.

[57] M. B. Thistlethwaite, *A spanning tree expansion of the Jones polynomial*, Topology **26** (3) (1987), 297–309.

[58] M. B. Thistlethwaite, *Links with trivial Jones polynomial*, J. Knot Theory and its Ramifications **10** no. 4 (2001), 641–643.

[59] W. Thomson, *On vortex motion*, Trans. Roy. Soc. Edinburgh 25 (1867), 217–260.

[60] H. F. Trotter, *Noninvertible knots exist*, Topology 2 (1964), 275–280.

[61] P. Vogel, *Representation of links by braids: a new algorithm*, Comm. Math. Helv. **65** (1990), 104–113.

[62] J. Weeks, *SnapPea: a computer program for creating and studying hyperbolic 3-manifolds*, computer software, http://www.geometrygames.org.

[63] J. Weeks, *Convex hulls and isometries of cusped hyperbolic 3-manifolds*, Topol. Appl. **52** (1993), 127–149.

[64] J. Weeks, *Computation of hyperbolic structures in knot theory*, this volume.

[65] W. Whitten, *Knot complements and groups*, Topology 26 (1987), 41–44.

[66] S. Yamada, *The minimal number of Seifert circles equals the braid index of a link*, Invent. Math. **89** (1987), 347–356.