The ? Onteora Club founded 1887

See first script above

The Bushes
Onteora Club
Tannersville, N.Y.
Aug 31, 19

Dear Prof Veblen,

I have just received your letter and am writing in hopes of clearing up the point you raise.

First, let me recall the surfaces S with which we have to deal. If we are working modulo 2, S is the image of a complex K^* a system of complexes, made up of un-sensed cells. The boundary of K consists of such one-cells as occur on the boundary of an odd number of 2-cells. If we are working with the Poincaré system, S is the image of a complex K made up of sensed cells. The boundary of K consists of the one cells of K each counted $\alpha - \beta$ times

where α is the number of times the one-cell occurs in the boundary of a 2-cell so as to point in the same direction as the positive direction along the boundary and β is the number of times it occurs on a boundary so as to point in the negative direction. The complex S is not completely determined until we have sensed all its 2 cells. It is closed if $\alpha - \beta = 0$ for all the one cells. It is two-sided in the sense that if we describe a closed path in S which does not cross the boundary of S, the indicatrix returns unchanged ~~then we get~~ back to the starting point.

Now to take up the particular point that you raise:

You refer to § 6 where we have a curve $\underline{c}$ of the manifold [illegible], with a curve $\underline{d}$ of the polyhedron π forms the complete boundary of a surface S.

[By the way, you speak of S in your letter as the image of a <u>non-singular surface</u>. It is merely the image of a <u>complex</u>]. If <u>d</u> occurs twice, (the case you raise), then d is not of the boundary of S if we are working modulo 2. We therefore have $c \sim 0 \pmod 2$, which is an example of the case mentioned in the next to the last sentence, second par., §6.

If we are dealing with the Poincaré numbers, the sense of the 2-cells making up S is determined by the sense of the curve <u>c</u>. ~~The 2-cells in turn~~ If <u>d</u> occurs on the boundary of one set of 2-cells in the positive sense and once in the negative sense, we have again $c \sim 0$. If d occurs twice in the same sense,

$$c \sim 2d \text{ or } -2d$$

depending on our choice of the positive direction along d.

In the projective plane, a projective line is homologous to 0, (à la Poincaré) since it bounds a 2-cell when counted twice. <u>It is not homologous to 0, modulo 2.</u> There is one coefficient of torsion 2 in the Poincaré theory.

I don't see as yet any criticism of ~~—~~ §15 of our joint paper (Annals June 1913) giving the relation between the numbers R, P, and coefficients of torsion.

Have just returned from a flying visit to N.Y. and Princeton to look up some matters in the literature of Anal. Sit. Was looking over Tait on knots among other things. He really doesn't get very far. He merely writes down all the plane ~~—~~ projections of knots with a limited number of crossings, Colored Blue in original letter

tries out a few transformations that he happens to think of and assumes without proof that if he is unable to reduce one knot to another with a reasonable number of tries, the two are distinct. His invariant, the generalization of the Gaussian invariant

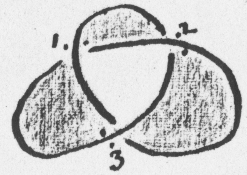
$$\iint \frac{(x'-x)\,dy\,dz' - dz\,dy' + \;-\,-\,-\,-}{\left[(x'-x)^2+(y'-y)^2+(z'-z')^2\right]^{3/2}}$$

for links is an invariant merely of the particular projection of the knot that you are dealing with, — the very thing I kept running up against in trying to get an integral that would apply. The same is true of his "beknottedness".

* Here is a genuine and rather jolly invariant:

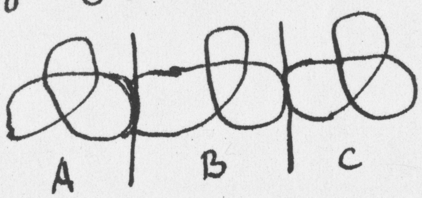
Take a plane ~~section~~ projection of the

knot and color alternate regions light blue (or if you prefer, baby pink). Walk all the way around the knot and every time you go over a crossing, put a dot to either side of the curve just beyond the crossing.



Form a square matrix with one column to each vertex and one row to each region, except that you leave out one arbitrary blue region and one arbitrary white one (most ~~arbitrarily~~ conveniently the outside one). Put a 0 in the ith row and jth column if the jth vertex is not on the boundary of the ith region, a +1 if the jth vertex is on the boundary of the ith region and if there is a dot in the corner of that region corresponding to this vertex, Put a -1 if the vertex is on the boundary but there is no dot in the , corner.

The elementary divisors of the matrix are invariants of the knot (or link; the thing goes equally well for links). The number of even elementary divisors is one less than the number of distinct curves of the system. If we build one system up by piecing on other systems A, B, C, the determinant of the resultant system is the product of the determinants of the individual ones.



A B C

More about this and other things when we meet. Am waiting for your book with impatience.

Sincerely

Alexander

P.S. The trefoil knot has the invariant (3), the Bear design:  the invariants (4,4). There are at least several errors in Tait's classification.

Postscript

I forgot to answer your second point about the possibility of the cells $A_1A_2A_3$ giving the cell $B_1B_2B_3$ over and over again. (p 152, second par). — Let the cell $B_1B_2B_3$ occur over and over again if ~~[illegible]~~ it wants to. A complex of π may contain a cell more than once, in which case it must be regarded as a multiple cell. If you are working modulo 2, you may cancel off coincident cells in pairs (if you feel like it); if you are working à la Poincaré, you may cancel off the coincident but oppositely sensed cells, but not similarly sensed ones. For certain complexes π, it is necessary to count cells more than once for the Poincaré theory to be valid.

J. W. A.