JIM HOSTE, MORWEN THISTLETHWAITE, AND JEFF WEEKS

The First 1,701,936 Knots

The history of knot tabulation is long established, having begun over 120 years ago. In many ways, the compilations of the first knot tables marked the beginning of the modern study of knots, and it is perhaps not surprising that as knot theory and topology grew, so did the knot tables. Over the last few years, we have extended the tables to include all prime knots with 16 or fewer crossings. This represents more than a 130-fold increase in the number of tabulated knots since the last burst of tabulation that took place in the early 1980s. With more than 1.7 million knots now in the tables, we hope that the census will serve as a rich source of examples and counterexamples and as a general testing ground for our collective intuition. To this end, we have written a UNIX-based computer program called *KnotScape* which allows easy access to the tables.

The account of our methodology is prefaced by a brief history of knot tabulation, concentrating mostly on events taking place within the last 30 years. The survey article [Thi1] contains further details on the work of the nineteenth-century tabulators, but, above all, the reader is encouraged to consult the original sources, in particular the excellent series of papers by Tait [Tai]. Kirkman's papers make fascinating reading, as they abound with original ideas and ornate language—his definition of the term "knot' is a single sentence of 101 words. Conway's landmark paper [Con] is also highly recommended.

An important feature of our project is that we have worked in two completely separate teams, producing two tabulations which were kept secret until after they were complete.[1] Although it would be foolhardy to claim with absolute certainty that our tables are correct, we must report the gratifying experience of finding that our lists of 1,701,936 knots were in complete agreement! Moreover, we did not use exactly the same methods, the primary difference being the use of hyperbolic geometry by Hoste and Weeks and the complete absence of hyperbolic invariants in Thistlethwaite's approach. Nevertheless, our overall programs are similar in spirit and differ little from the methods of most tabulators who precede us. As part of our tabulation, using Weeks's program *SnapPea* we were able to compute the *symmetry groups* of the knots; we have included a short introduction to this beautiful and intriguing topic.

[1]At a recent conference, people who were aware of our project jokingly scolded us for conversing together.

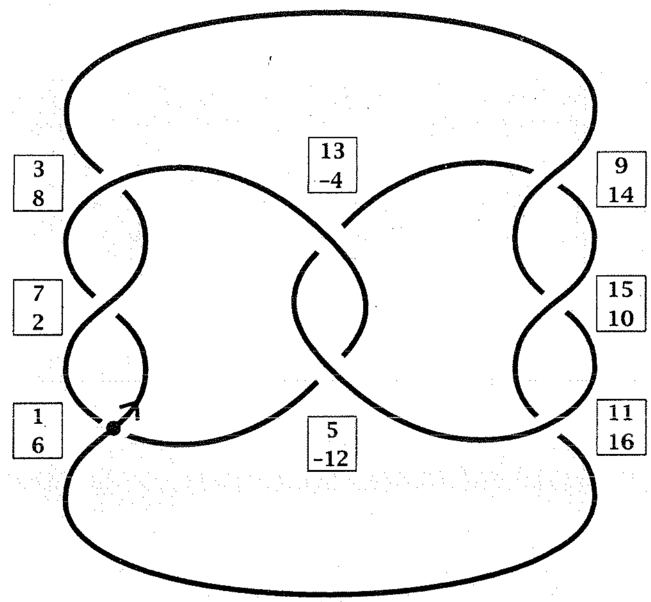
Obviously, we now have a great deal of data, and reporting on every aspect of the tabulated knots is not possible. Instead, in the first of three appendices we present a statistical summary of the census. The second appendix contains lists of hyperbolic knots with selected symmetries, and the final appendix contains brief descriptions of *KnotScape* and Weeks's program *SnapPea* which figures prominently in our work. We describe the hardware and software requirements of these programs, where to obtain them, and their capabilities.

A Brief History of Knot Tabulation

In the late 1860s, the great Scottish physicist William Thomson (Lord Kelvin) suggested that atoms were knotted vortices in the ether. If only we could better understand knots, we could unravel the secrets of the atom and of matter itself! Inspired by this theory, Thomson's countryman and fellow physicist, Peter Guthrie Tait, embarked on a major investigation of knots which included production of the first knot tables. By a *knot* we mean a smoothly embedded circle in 3-dimensional Euclidean space $\mathbb{R}^3$. A knot *diagram* is a projection of a knot into a plane containing only transverse double points and, furthermore, drawn with *crossings* at each double point so that the embedding can be recovered from the diagram (Fig. 1). We will consider different knots to be equivalent if there is a homeomorphism of $\mathbb{R}^3$ to itself taking one knot to the other. Thus, a single knot can be represented by infinitely many diagrams, but only a finite number of diagrams will have a minimal number of crossings, and it is with respect to this *crossing number* that Tait organized his table. The trivial knot, or unknot, can be drawn with no crossings, the trefoil knot with three, the figure-eight knot with four, and so on (Fig. 2).

The strategy employed by Tait, and still used today in our tabulation, is simple: enumerate all possible diagrams up to a given crossing number and then group together those diagrams that represent the same knot type. To begin this process, Tait invented a scheme for encoding knot diagrams. Many years earlier, Gauss and his student Listing had studied knots and invented their own notations for this purpose [Lis]. Although initially he was unaware of their work, Tait's scheme is similar. Our own notation, first used by Dowker and Thistlethwaite [DT, Thi1], is a further refinement. It allows any knot diagram with N crossings to be encoded as a sequence of N (signed) even integers $a_1, \dots, a_N$, where the sequence of absolute values is a rearrangement of $2, 4, \dots, 2N$. The encoding scheme is described in Figure 1, and its subtleties and limitations are discussed in the third section.[2]

Tait considered all such sequences up to seven crossings and successfully grouped them together by knot type. In 1876, he published his first table, containing the knots through seven crossings and all their minimal diagrams. (Figure 2 illustrates these 15 knot types in the order in which they are listed in our table supplied with the software package *Knotscape, viz.* Appendix III). But, daunted by the combinatorial explosion of sequences for larger crossing number, Tait stopped at seven crossings. It is important to remember that Tait had no theorems from topology to enable him to distinguish different knots. In fact he wrote, "... though I have grouped together many widely different but equivalent forms, I cannot be *absolutely* certain that all those groups are essentially different one from another." Indeed, it is the task of grouping the diagrams together by knot type rather than enumerating all possible diagrams that remains to this day the most difficult part of knot tabulation, for producing all possible diagrams is algorithmic and therefore, at least theoretically, trivial. However, for a large crossing number, the sheer number of possible combinations is so huge that



1	3	5	7	9	11	13	15
6	8	−12	2	14	16	−4	10

Figure 1. To encode a diagram, choose a basepoint and orientation of the knot; in the above figure, the chosen basepoint is indicated by a black dot at the overpass labeled 1. Traveling from this basepoint in the given direction, label points on the knot curve lying directly above or below crossings with consecutive integers 1, 2, 3, ... Each crossing thus receives two labels, one even and one odd, and this defines a one-to-one correspondence between the set of odd labels and the set of even labels. The overcrossing–undercrossing structure is then captured by associating a minus sign to each even integer which is the label of an overpass; thus, a diagram is alternating (i.e., –over–under–over–under–) if and only if all even integers have the same sign. The data for this example are displayed in the table immediately below the figure; because the odd numbers have been written in their natural order, all the information is contained in the sequence of signed even numbers. Therefore, for the given choice of basepoint and direction, the code for this diagram is 6 8 − 12 2 14 16 − 4 10. We choose the *standard* code for the diagram to be the sequence which is minimal over all choices of starting point and direction, with respect to a suitable ordering of sequences.

[2]A mild refinement of this notation [DH] can be used to encode link diagrams. (*A link* of k components is the union of a family of k disjoint simple closed curves in $\mathbb{R}^3$.)

even with today's high-speed computers, the task of enumerating all possible diagrams remains difficult in practice.

To aid in the comparison of different diagrams, Tait invented a certain diagrammatic transformation which preserves crossing number, now known as the *flype* (Fig. 3).[3] He also classified crossings as *left-handed* with associated sign -1, or *right-handed* with associated sign $+1$ (Fig. 4); the *writhe* of the diagram[4] is then defined as the sum of the signs of the crossings. He further declared that a crossing is *nugatory* ("worthless") if there is a circle in the projection plane meeting the diagram transversely at that crossing, but not meeting the diagram at any other point (Fig. 4). Nugatory crossings can obviously be removed by twisting, so they cannot occur in a diagram of minimal crossing number. A diagram is *reduced* if none of its crossings is nugatory.

Tait set forth a number of conjectures concerning alternating knots, none of which was resolved until the advent of the Jones polynomial in 1984. He conjectured (i) that reduced alternating diagrams had minimal crossing number, (ii) that any two reduced alternating diagrams of a given knot had equal writhe, and (iii) that any two reduced alternating diagrams of a given knot were related via a sequence of flypes. The third of these conjectures implies the second, because flypes preserve writhe. The first two conjectures have been proved in various ways [Kau, Mur1, Thi2, Mur2, Thi3], but all proofs use properties of the Jones polynomial or the Kauffman two-variable polynomial. A solution of the third conjecture is given in [MT]; the proof is mostly geometric, but, again, it relies in an essential way on properties of the Jones polynomial. The confirmation of these conjectures has significantly lightened the task of tabulating alternating knots.

After Tait's first paper appeared, he learned of the work of the Reverend Thomas P. Kirkman [Kir1, Kir2] who had himself set out to enumerate knot projections. Kirkman had used a method quite different from Tait's; he started with a relatively small set of "irreducible" projections and then produced complete lists of knot projections by inserting crossings in a systematic way. Nearly a century later, Conway used a modification of Kirkman's method with great success [Con].

Using Kirkman's projections Tait went on to produce tables, in 1884 and 1885, of alternating knots (and all their minimal diagrams) through 10 crossings. Just before going into print, Tait learned of another census of knots through 10 crossings produced by the American C.N. Little [Lit1]. Comparing their work, Tait noted one duplication in his own table and one duplication and one omission in Little's, and promptly corrected his own table prior to publication.

With Tait's encouragement, Little went on to tabulate the 11-crossing alternating knots, starting from the polyedral[5] (*sic*) diagrams of Kirkman [Lit3]. Little also undertook the more difficult task of tabulating the nonalternating knots, ones which admit no alternating diagram. These

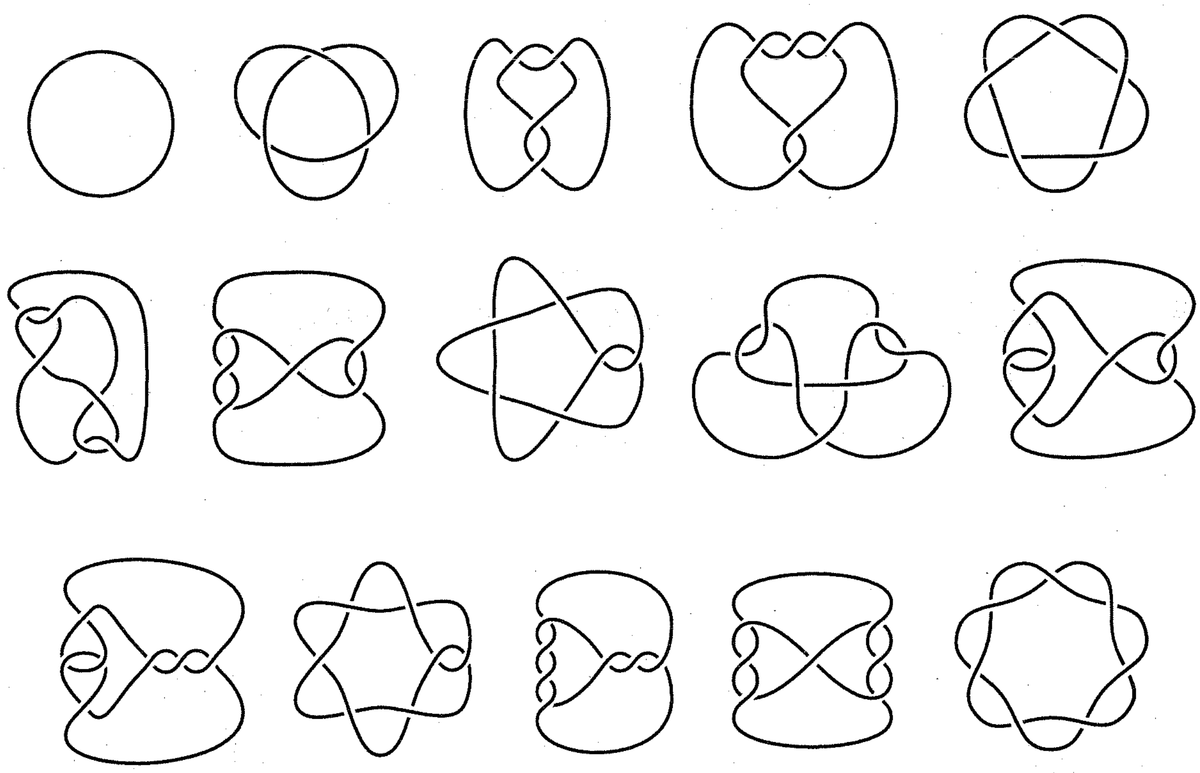


Figure 2. Knots to seven crossings.

[3]Tait had used the word flype to denote a different kind of transformation, namely a change of infinite complementary region: "flype" is an old Scottish verb whose approximate meaning is "to turn or fold back" (as with a sock). Currently, the word "flype" designates the transformation illustrated in Figure 3.

[4]Tait and Little used the term "twist" in place of "writhe."

[5]Kirkman held passionate views on the spelling of certain words.

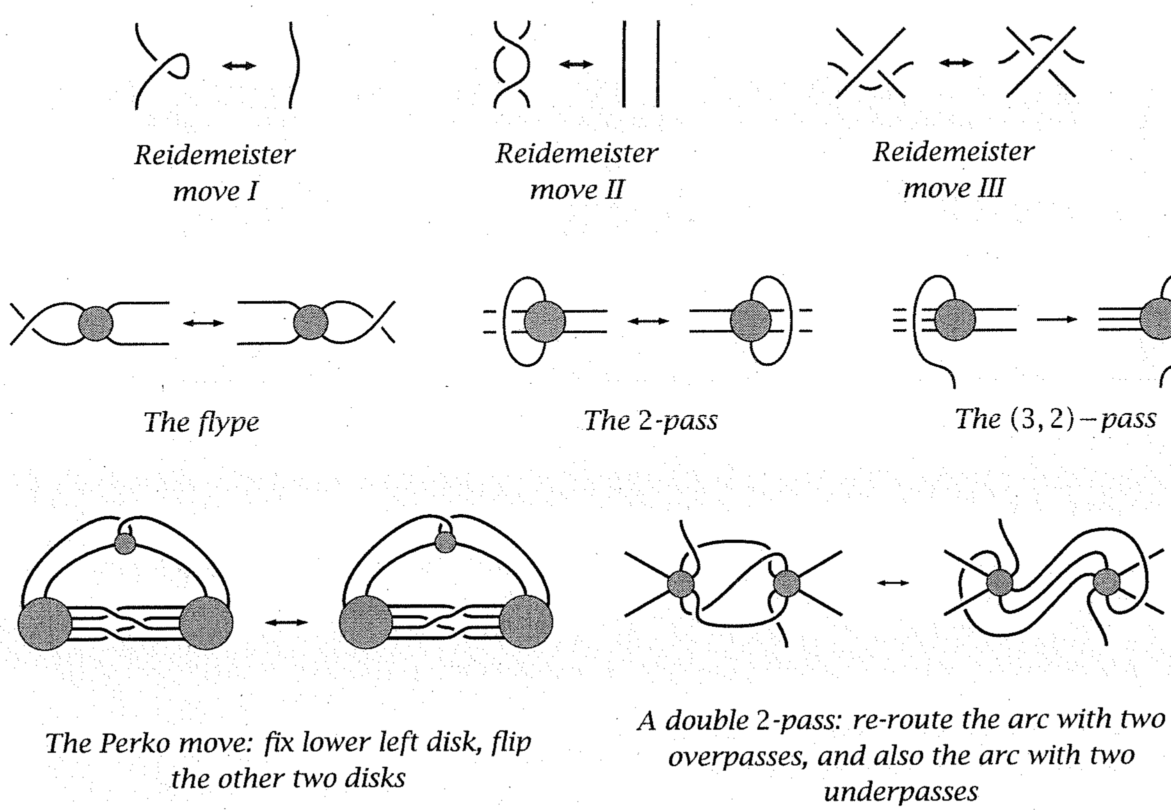


Figure 3. Moves on diagrams.

do not appear with fewer than eight crossings, and from Tait's first paper, it is evident that initially he did not believe that nonalternating knots were possible. In fact, the first *proof* of the existence of a nonalternating knot did not appear until 1930. Little states that he worked for 6 years, from 1893 to 1899, to produce his list of 43, 10-crossing nonalternating knots [Lit4]. As we shall soon see, his list had no omissions, but it did have one duplication.

One obstacle to tabulating nonalternating knots is their sheer quantity. Although nonalternating knots do not predominate until 13 crossings (as mentioned earlier, they do not even appear until 8 crossings), it is plausible that the proportion of knots which are alternating tends exponentially to zero with increasing crossing number. Recently, this was proved for links by Sundberg and Thistlethwaite [ST, Thi4]. Determining the asymptotic rate of growth of the number of knots is an interesting problem [ES]; it is known [Wel, Thi4] that if K_n denotes the number of n-crossing prime knot types, then $\lim\sup(K_n)^{1/n} < 13.5$.

Another problem with nonalternating diagrams is that flypes no longer suffice to pass between all minimal diagrams of the same knot. Although this was apparent to Little, he erroneously believed that just two kinds of moves, the flype and the 2-*pass*, were sufficient.[6]

Finally, after over 25 years of laborious handwork, Tait, Kirkman, and Little had created a table of alternating knots through 11 crossings and nonalternating knots through 10 crossings. Of course, in the absence of a rigorous theory, they could not know whether their tables were correct; indeed, a few errors have come to light in the ensuing years. But, remarkably, the table of alternating knots through 10 crossings has stood the test of time.

The era of rigorous knot theory began in the early part of this century. In 1914, the subject of topology had developed to the extent that Dehn was able to publish a proof that the right-handed and left-handed trefoils were distinct [Deh]. In 1927, using the first homology groups of branched cyclic covers, Alexander and Briggs were able to distinguish all the tabulated knots through nine crossings, with the exception of three pairs [AB]. In 1932, Reidemeister completed the classification of knots up to nine crossings, using the linking numbers of branch curves in irregular covers associated to homomorphisms of the knot group onto dihedral groups [Rei].

In 1949, Schubert proved that every knot can be uniquely decomposed, up to order, as a *connected sum* of *prime* knots (Fig. 4). In close analogy with arithmetic, a knot is *prime* if it cannot properly be decomposed as a connected sum. In the light of Schubert's theorem, it is only necessary to tabulate prime knots; the composite knots are then easily constructed by taking connected sums.

Another important consideration is that of *chirality*. So far, we have considered two knots to be equivalent if there is a homeomorphism of $\mathbb{R}^3$ mapping one to the other. According to this definition, any knot and its mirror image (with respect to some plane) are equivalent. But this does not tally with the layman's concept of equivalence; a piece

[6]Thistlethwaite used no fewer than 13 different diagrammatic moves when generating the initial raw list of 16-crossing nonalternating knots, yet this list of 1,018,774 knots still had 9,868 duplicates. Two of the more exotic moves are illustrated in Figure 3.

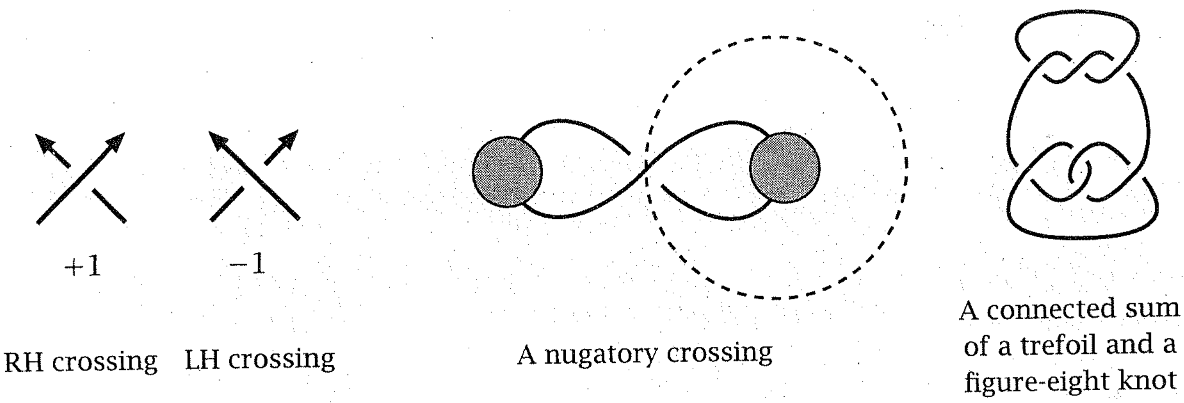


Figure 4

of rope tied as a left-handed trefoil cannot be manipulated into a right-handed trefoil. It would be more intuitive to consider knots K_1 and K_2 to be equivalent if they are related by an *ambient isotopy*; by this we mean that there exists a continuously parametrized family of homeomorphisms h_t: $\mathbb{R}^3 \to \mathbb{R}^3$ $(0 \le t \le 1)$ such that h_0 is the identity map on $\mathbb{R}^3$ and h_1 maps K_1 onto K_2. One can think of t as representing time, and one can imagine the knot moving in a viscous fluid; the isotopy describes how to move K_1 to K_2 and how the molecules of fluid are moved in the process. At first, this new definition seems to be radically different from the original one, but, in fact, it amounts merely to saying that two knots are equivalent if there is an *orientation-preserving* homeomorphism of $\mathbb{R}^3$ mapping one to the other. Under this stronger version of equivalence, which knots remain equivalent to their mirror images? Those that do are endowed with a special kind of symmetry and are called *achiral* or *amphicheiral*.

When we discuss amphicheirality in more detail later in the article, we shall use the following alternative definition: a knot K is amphicheiral if there exists an *orientation-reversing* homeomorphism of $\mathbb{R}^3$ mapping K to itself. To see that the definitions are equivalent, simply compose the homeomorphism of $\mathbb{R}^3$ given by either definition with reflection in the projection plane.

Amphicheiral knots are important to chemists, who are often concerned with the right- or left-handedness of molecules. The figure-eight is amphicheiral, a fact that was known to Listing. The trefoil certainly *appears* not to be equivalent to its mirror image, but without proof, this cannot be asserted as a fact. As mentioned earlier, the first proof was given by Dehn in 1914 [Deh], in the early days of topology.

Tait was interested in the concept of amphicheirality but did not have an intrinsic definition; he was somewhat hampered by being tied to the projection plane and had rather artificial distinctions between "kinds" of amphicheirality. Nonetheless, he successfully identified all amphicheiral alternating knots with up to 10 crossings; he did not consider nonalternating knots, but, as it happens, there are no amphicheiral nonalternating knots with fewer than 12 crossings. He conjectured that amphicheiral knots with odd crossing numbers could not exist; this is now known to be the case for alternating knots, but our census has turned up a 15-crossing nonalternating amphicheiral knot (Fig. 5).

Another way of refining the notion of equivalence is to consider the orientation, or direction, of the knot curve. A knot is said to be *invertible* (or *reversible*) if there is an ambient isotopy carrying the knot onto itself, but with its direction reversed. The trefoil is clearly invertible: take the diagram illustrated in Figure 2 and rotate it through half a turn about an axis in the plane. In fact, every knot illustrated in Figure 2 is invertible, and it is not immediately clear that there exists a knot which is not invertible. This question was not resolved until 1964, when Trotter discovered an infinite family of noninvertible knots, beginning at nine crossings [Tro]. The situation with a low crossing

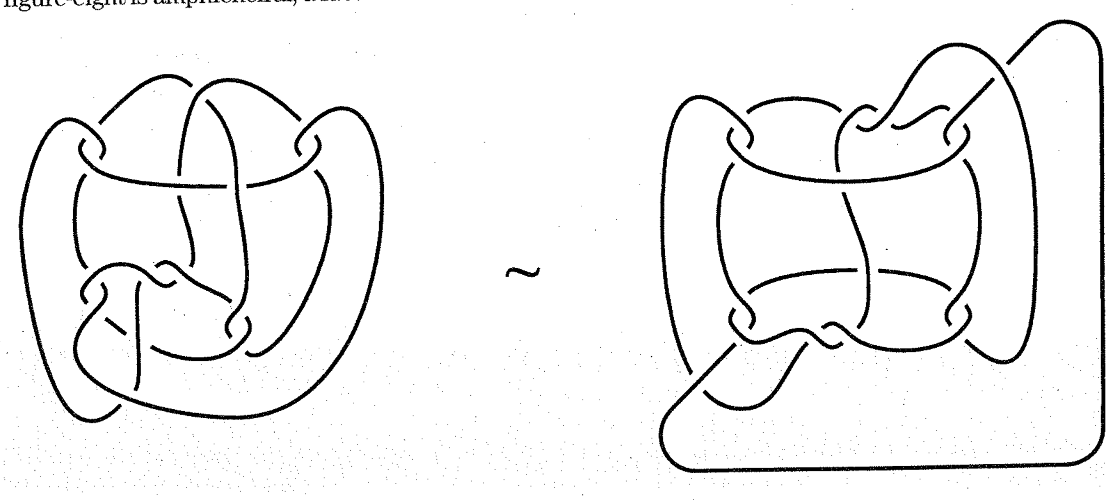


Figure 5. The 15-crossing amphicheiral knot.

number is not typical, however; inspection of Table A1 in Appendix I reveals that, overall, only a tiny proportion of the knots up to 16 crossings are invertible.

It is obviously of fundamental importance to be able to show that the knots listed in any purported table of prime knots really are prime. It is also desirable to know which knots are amphicheiral or invertible.[7] During the course of several decades, researchers devoted much effort to answering these and other questions for knots tabulated in the nineteenth century, but for over 50 years, almost nothing was done to verify or extend the tables. The only exception to this was a compilation of 12- and 14-crossing amphicheiral knots undertaken by Haseman in 1917 and 1918 [Has1, Has2].

Activity picked up again in the 1960s, when Conway invented a striking new notation for knots and links. This notation, which uses some of Kirkman's ideas, enabled Conway to enumerate all prime knots up to 11 crossings and all links up to 10 crossings. Conway's scheme was so efficient that he claims to have completed this task, by hand, in a "few hours"! Conway found 11 omissions and 1 duplication in Little's table of 11-crossing alternating knots, but his own new table of 11-crossing nonalternating knots had 4 omissions. Conway also overlooked the famous duplication in Little's table of 10-crossing nonalternating knots; this duplication was finally brought to light in 1974 by Perko, who showed that the two diagrams were related by the move which bears his name (Fig. 3).[8] This pair of 10-crossing nonalternating diagrams of the same knot have different writhes and this is probably why the duplication went undetected for so long. Indeed, Little had even published a "proof" that the writhe of a minimal diagram was a knot invariant, based on the mistaken assumption that flypes and 2-passes sufficed to pass between any two minimal diagrams of the same knot.

In the late 1970s, Caudron [Cau] used an alternative version of Conway's notation to retabulate knots up to 11 crossings, discovering, in the process, the 4 omissions referred to above. Meanwhile, Bonahon and Siebemann, noticing that Conway's notational system reflected deep structure properties of knots, proved a general classification theorem for the family of so-called *algebraic* knots [BS]. The great majority of knots through 11 crossings are algebraic, although (as with all "nice" families) they are soon overwhelmed by the nonalgebraic knots. In a *tour de force*, Perko [Per 2, Per 3] computed enough invariants by hand to distinguish the knots not covered by Bonahon and Siebenmann's result, thus finally completing the classification through 11 crossings. These efforts mark the end of the hand-calculation era, as all subsequent tabulations have been carried out by computer.

In the early 1980s, Dowker and Thistlethwaite computerized the tabulation process and extended the table to 13 crossings. Although Conway's notation has been a major conceptual influence in knot theory and is very compact for knots of low crossing number, it does not lend itself readily to computer programming, as it draws on a large set of symbols assembled according to a rather large set of rules, both of which grow with crossing number. Instead, Dowker and Thistlethwaite used the refinement of Tait's notation already mentioned to enumerate all possible diagrams. Flypes, 2-passes, and other moves (Fig. 3) were used to group diagrams into equivalence classes which could then be distinguished by topological invariants.

The table stood at 13 crossings for about a decade until Hoste was recruited by local high-school students who had just won access to a Cray supercomputer. Together, they tabulated all alternating knots through 14 crossings and provided the first check of Thistlethwaite's list of alternating knots [Arn]. At the same time, Thistlethwaite returned to the tabulation problem and extended the table even further. Most recently, Hoste and Weeks have collaborated to produce a table. Working in parallel with Thistlethwaite, we have currently tabulated all prime knots through 16 crossings and, as this article goes to press, have begun the 17-crossing list.

Our Methodology

Both our tabulations, Hoste/Weeks's and Thistlethwaite's, begin by listing all prime alternating knots of a given crossing number. We do this by generating all possible alternating diagrams and then grouping them into flype equivalence classes. We both use the same encoding scheme for diagrams, namely the sequence of even integers already explained in Figure 1. We will refer to these sequences as *DT sequences* (for Dowker and Thistlethwaite). Using the same notation makes it easy for us to compare our results.

To reach our immediate goal, namely a list of all N-crossing alternating knot types, we need to be aware of certain features of the DT encoding scheme. Our first observation is that the encoding of any particular diagram depends on the choice of basepoint and direction. There are $(2N)(2) = 4N$ such choices, and for highly asymmetric diagrams, these choices can result in $4N$ different sequences. We need to choose a distinguished member of the collection of sequences representing a given diagram; to this end, we declare that the *standard sequence* for the diagram is the DT sequence, which is minimal with respect to lexicographical ordering, over all choices of basepoint and direction. Indeed, we may go further and declare the standard sequence for a *knot type* to be the minimal sequence over all diagrams of minimal crossing number representing that knot type.

A second point to consider is that not every DT sequence encodes a knot diagram. A moment's reflection should convince the reader that the objects really being encoded by DT sequences are 4-valent graphs obtained by identifying pairs of points on a circle, and, clearly, such a graph need not be planar. Thus, it is not surprising that some DT sequences are not *realizable*. However, algorithms for deciding the planarity of a graph are well known and it is a

[7] For hyperbolic knots, *SnapPea* easily settles all these questions.

[8] To his credit, Conway had discovered more complicated instances of the Perko move among the 11-crossing knots.

simple matter to test a DT sequence for realizability. Furthermore, Dowker and Thistlethwaite show that if a DT sequence is coming from a *prime* diagram, then the diagram is uniquely determined by the sequence, up to reflection and isotopy of the extended plane [DT]. (A diagram is prime if there is no circle in the plane cutting the diagram transversely in two points, with crossings lying on both sides; an example of a diagram which is not prime is that of the connected sum of the trefoil and the figure-eight given in Figure 4.)

As already stated, we are only interested in prime knots and must, therefore, exclude DT sequences representing composite knots. In general, deciding whether a given knot is prime is a nontrivial matter, as the decomposing 2-sphere of a composite knot might not be immediately visible. Fortunately, in the case of alternating knots, there is a beautiful result of Menasco [Men] that can be brought to bear. The result states that a reduced alternating diagram represents a prime knot if and only if the diagram is itself prime; informally, an alternating knot is prime if and only if "it looks as if it is prime."[9] Determining whether a DT sequence represents a prime diagram is straightforward: a diagram is composite if and only if there is a proper subinterval of $\{1,2,\ldots,N\}$ such that each number in the subinterval is paired with another number in the same subinterval (the subinterval contains the points associated with crossings of one of the summands of the diagram). Note that if a diagram contains a nugatory crossing, then it also fails to be prime; therefore, once we have eliminated the composite diagrams, we no longer have to worry about nonreduced diagrams.

Our strategy is now clear. We generate all possible DT sequences of length N in lexicographic order and immediately discard those that are nonreliazable and those that are nonprime. We also discard those sequences which are not minimal over all choices of starting point and direction. After this filtering process, we are left with a list of sequences representing the prime, N-crossing alternating diagrams. These sequences are then sorted into equivalence classes with respect to the operation of flyping; for this step, it is necessary to write a procedure which detects all possible flypes and implements them as transformations of DT sequences. The final list of alternating knot types is then obtained by recording the member of each equivalence class which is lexicographically minimal. In practice, of course, it is not necessary ever to have the complete "raw" list of diagrams; one can reject diagrams that are not minimal with respect to flyping, as they are generated.

The entire process is algorithmic and straightforward to implement as computer code. Nonetheless, we should mention that this is a nontrivial exercise in computer programming, given that we wish the entire task to be completed in a reasonable amount of time! Our runtimes for $N = 16$ are presently of the order of 1–2 weeks,[10] and our own experience shows that careless programming can easily increase this by one or two orders of magnitude. The trick is to rule out whole sets of sequences whose first few entries already guarantee eventual failure of one of the tests. For example, we may ignore DT sequences that start with 2, as the corresponding diagram would have a nugatory crossing labeled $\{1, 2\}$. Also, if a sequence begins with 4, primality of the associated diagram dictates that the numbers paired with 2 and 3 must differ by 1. Furthermore, any sequence of length >3 which begins 4 6 2 . . . cannot represent a prime diagram, as there will be a trefoil summand. These examples are quite simple, but hint at the possibilities.

After obtaining the list of all prime, alternating, unoriented knots with N crossings, we generate the nonalternating knots with N crossings. Each N-crossing nonalternating diagram can be obtained by switching crossings of an N-crossing alternating diagram, so by switching crossings in all possible ways in each N-crossing alternating diagram, we produce all possible N-crossing nonalternating knot types. At first, it might appear that we need to use all (reduced) alternating diagrams, but, in fact, we only need to use one alternating diagram per alternating knot type. For if D and E are two alternating diagrams related by flypes and E' is a nonalternating diagram obtained by switching crossings of E, then E' is related by flypes to a nonalternating diagram D' obtained by switching crossings of D.

To generate the nonalternating diagrams, we just insert minus signs in all possible ways into the DT sequences that represent the alternating knots. Because we are only interested in tabulating knots up to reflection, we may assume the first entry of each sequence is positive. Thus, there are 2^{N-1} sequences to consider for each alternating diagram. Most of these nonalternating diagrams reduce to diagrams of fewer crossings; again, with careful programming, we can avoid ever considering the myriad sequences that immediately reduce to smaller diagrams by means of a single Type II Reidemeister move (Fig. 3) or even a combination of two or three Reidemeister moves. Still, many diagrams are left, and a more careful sorting into knot types must be undertaken.

Before continuing with this discussion, we describe how we have chosen to extend the lexicographical ordering of unsigned DT sequences of length N to signed DT sequences of length N, for, again, our ultimate aim is to settle on a unique representative for each knot type. Our first convention is that a DT sequence of positive numbers always precedes a sequence containing one or more negative numbers; this merely reflects the fact that we enumerate all alternating diagrams before proceeding to the nonalternating diagrams. Now, suppose that we have two sequences s_1 and s_2, each with at least one negative term. For $i = 1, 2$, let $|s_i|$ be the sequence whose terms are the absolute values of those of s_i. If $|s_1|$ precedes $|s_2|$ lexicographically, we declare that

[9] The reader can easily construct examples of (nonalternating) prime diagrams representing composite knots, and also examples of composite diagrams representing prime knots; such examples illustrate the essential difference between the two concepts "prime knot" and "prime diagram" and highlight the significance of Menasco's result.

[10] Each increment of crossing number results (roughly) in a 10-fold increase in computer runtime. Our programs will accomplish Conway's enumeration up to 11 crossings in about 9 s!

$s_1 < s_2$. If, on the other hand, $|s_1| = |s_2|$, we look at the first position where s_1 and s_2 have terms of opposite sign and declare that $s_1 < s_2$ if s_1 has a negative term in that position.

It is at this point that our two methods of tabulation begin to differ significantly, as Hoste and Thistlethwaite apply different types of moves to the nonalternating diagrams in an effort to eliminate redundant diagrams. After being left with nonalternating diagrams that do not obviously reduce to fewer crossings, Hoste considers all diagrams related to a given one by flypes and 2-passes and in every diagram in this class looks for any reduction to fewer crossings given by an (i, j)-pass, where $i > j$ (Fig. 3). If none is found, the lexicographically smallest diagram in the class is retained. What is left in the end is a superset of the set of all unoriented prime nonalternating knots with N crossings, up to reflection. Even for $N = 10$, this superset is too big: the famous Perko pair of diagrams mentioned earlier still remains and the list contains 43 rather than 42 diagrams. For $N = 16$, the list has approximately 10% too many diagrams. At this point, no further attempts are made by Hoste to eliminate duplicates by means of Reidemeister or other diagrammatic moves. Instead, the list is passed to Weeks and his computer program *SnapPea*. We will return in a moment to a discussion of how the final list of nonalternating knots is found by *SnapPea*.

Thistlethwaite, on the other hand, applies more diagrammatic moves to each nonalternating diagram. In addition to the flypes and pass moves used by Hoste, he also employs "double-pass" moves, the "Perko" move (Fig. 3), and a few other esoteric moves designed specifically to root out stubborn pairs of equivalent diagrams. These moves preserve crossing number and, with just one exception, suffice to eliminate all duplicates through $N = 13$.

After arriving at an initial superset of nonalternating knots, Thistlethwaite then turns his attention to distinguishing as many knots as possible. He first computes the Jones polynomial, which places the knots in small equivalence classes, each class consisting of all knots with a given Jones polynomial. These equivalence classes are then attacked by invariants based on representations of the knot group (Perko had already used this type of invariant with notable success in dealing with 11-crossing knots). In the case at hand, a few thousand pairs and triples of knot diagrams still resolutely refused to be distinguished, but, fortunately, it was shown that the diagrams in each stubborn pair or triple were equivalent. The method for this last step was to apply moves to increase the number of crossings of the diagrams and then to apply all the previous moves to these "expanded" diagrams.

Returning to the superset of nonalternating diagrams generated by Hoste, the next step is the application of Weeks's program *SnapPea*. There are over 1.7 million knots in our table, but, amazingly, it turns out that all but 32 are *hyperbolic* and thus susceptible to the full weaponry of hyperbolic geometry (an explanation of the term "hyperbolic" follows shortly). Several important theorems now apply which lead, in the case of hyperbolic knots, to a complete knot invariant. This invariant is then used, in the case of the hyperbolic knots, to remove all duplicates from Hoste's list. The nonhyperbolic knots are so few in number that they are easily dealt with separately.

The first important theorem, due to Gordon and Luecke, is that two knots are equivalent if and only if their complements are homeomorphic [GL]. The second, due to Mostow and Prasad, states that if a knot complement admits a complete Riemannian metric of constant Gaussian curvature -1, in other words the knot is *hyperbolic*, then such a metric is unique [Pra]. Thus, two hyperbolic knots are equivalent if and only if their complements are isometric. The final result we need is the existence of a canonical triangulation of hyperbolic knot complements, shown to exist by Epstein and Penner [EP] and Sakuma and Weeks [Wks, SW]. The canonical decomposition is described in Figure 6. It is a complete invariant for hyperbolic knots, as two knot complements are isometric if and only if they have the same canonical triangulation. Finally, it is important to note that the canonical triangulation by ideal polyhedra can be described entirely combinatorially, by designating which faces of which polyhedra must be identified. Thus, once the canonical decomposition has been found for each of the hyperbolic knots on the list of nonalternating knots, they can be compared combinatorially. If two are alike, the knots are the same and the redundant diagram can be dropped from the list. If two are different, the two knots are different.

SnapPea takes as input Hoste's list of nonalternating diagrams and attempts to find the canonical decomposition for each. Through $N = 16$, the only nonhyperbolic knots are the 12 torus knots and 20 satellite knots listed at the end of this section.[11] For the rest, *SnapPea* succeeds in finding a hyperbolic structure, which it then uses to construct the canonical decomposition. Although the basic data used to describe the hyperbolic structure are algebraic numbers and may eventually be recorded as such by future versions of *SnapPea*, they are presently rounded off and stored as floating-point numbers. This has the undesired effect that roundoff error may lead *SnapPea* to a decomposition which is not the canonical one. To understand how this happens, recall the imagery of Figure 6. If two adjacent triangular faces of the convex hull are found to be coplanar to an accuracy of, say, 10^{-12}, should *SnapPea* treat them as distinct triangular faces, or should they be combined to form a single quadrilateral face? If *SnapPea* guesses wrong, it may give a false negative to the question, "Are these two hyperbolic knots the same?" Fortunately, false positives are impossible, because if two decompositions are equivalent, the knots must be the same, whether or not the decompositions are the canonical ones. Therefore, the list of knots computed by this method is guaranteed to be complete, but it could, in principle, contain duplications. Comparison with Thistlethwaite's results shows rigorously that, in fact, no duplications are present, because Thistlethwaite distinguishes knots by integer invariants.

We should point out that unlike the tabulation of the alternating knots, our methods for nonalternating knots are

[11]A famous theorem of W. Thurston states that *any* nonhyperbolic knot is either a torus knot or a satellite knot.

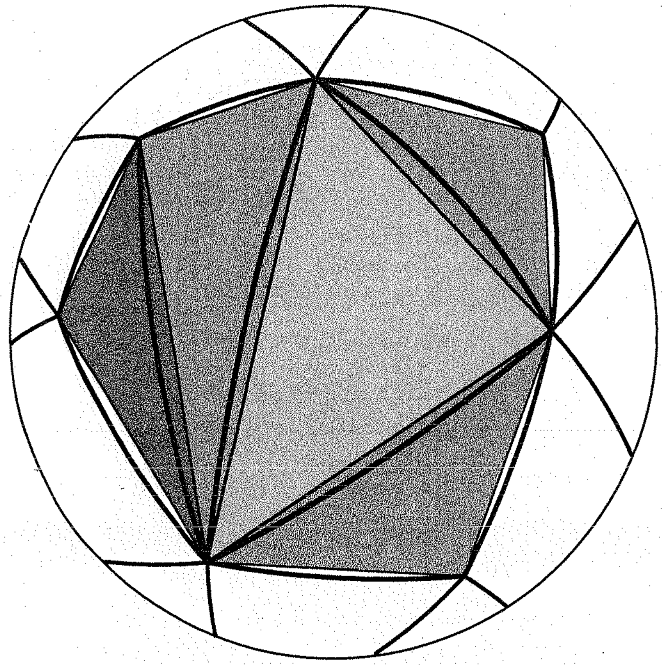


Figure 6. Given a fixed set of vertices, one may construct a canonical triangulation of a sphere by taking the convex hull in Euclidean 3-space and radially projecting its edges back onto the sphere itself. A similar construction in (3 + 1)-dimensional Minkowski space [EP, Wks, SW] defines a canonical decomposition for a hyperbolic knot complement.

not algorithmic. Instead, we simply employ a collection of methods that *work* for $N \leq 16$.[12]

After producing the two tabulations, it is then a simple matter to compare our results. Because we share the same encoding scheme and the same choice of lexicographic order, the two lists should be exactly the same. In practice, we have compared our lists after completing the table for each crossing number. Only for $N = 14$ did our results differ, and then only by one of us having 4 omissions that the other did not. A programming error was quickly found that accounted for the discrepancy. For $N = 15$ and 16, our lists have agreed on first comparison, a very satisfactory experience!

At this point, the reader might wonder how we can be certain that all the knots in our tabulation are prime. This is a nontrivial matter, but, fortunately, it is a fundamental property of hyperbolic knots that they cannot be composite. Therefore, in finding their hyperbolic structures, *SnapPea* has already established primality for all but 32 of the knots.[13] As mentioned earlier, each of the remaining knots is a torus or satellite knot. A proof that torus knots are prime may be found in a textbook on knot theory such as [BZ]; the satellite knots all have structures which are well known and documented (Fig. 9), and their primality follows from a simple geometric argument.

Therefore, we have produced a table of all prime, unoriented knot types up to N crossings, without duplications. So far, we have ignored the issues of amphicheirality and invertibility, but these issues need to be addressed if we wish to classify oriented knots up to isotopy. They are best discussed in the context of *knot symmetries*.

The Symmetry Group of a Knot

Recall that in plane geometry, a symmetry of a regular polygon of n sides is defined as a rigid motion (isometry) of the plane which maps the polygon onto itself; for example, a square may be mapped onto itself by any of four rotations about its center, by a reflection about a diagonal of the square, or by a reflection about a perpendicular bisector of opposite sides of the square. These eight symmetries of a square form a group under the operation of composition, known as the *dihedral group* D_4. More generally, a regular n-sided polygon has $2n$ symmetries, of which n are rotations and n are reflections, and these form the *dihedral group* D_n. Informally, there are $2n$ distinct ways of picking up the polygon and putting it back onto its original location. The n rotations also form a group, called the *cyclic group* Z_n.

In the theory of knots, symmetries are defined analogously. We could merely consider a *symmetry* of a knot K to be a homeomorphism of $\mathbb{R}^3$ which maps K to itself, or, more succinctly, a homeomorphism of the pair of spaces $(\mathbb{R}^3,K)$. However, it is natural to regard two symmetries of K as being equivalent if there is a continuously parametrized family of symmetries α_t $(0 \leq t \leq 1)$ such that one of the symmetries in question is α_0 and the other symmetry is α_1 (i.e., α_0 can be "deformed" to α_1 through the family of symmetries $\{\alpha_t\}$). Thus, it is customary to regard a symmetry of the knot K as an equivalence class of homeomorphisms of $(\mathbb{R}^3,K)$ with regard to this equivalence relation. As in plane geometry, the operation of composition induces a group structure on the set of equivalence classes. We can think of a symmetry of K as a way of transforming space so that the knot is mapped onto itself; we just have to remember that two symmetries are "the same" if one can be deformed to the other, as above.

We may classify knot symmetries into four types, depending on whether the symmetry reverses the orientation of $\mathbb{R}^3$ or reverses the orientation of K, according to Table 1.

In Figure 1, for example, if we perform a rotation through half a turn about a west–east axis in the projection plane

Table 1. Classification of knot symmetries.

Symmetry type	Orientation of $\mathbb{R}^3$	Orientation of K
0	Preserves	Preserves
1	Preserves	Reverses
2	Reverses	Preserves
3	Reverses	Reverses

[12]Of course, the lack of an algorithm does not detract from the rigor of our approach. We merely run the risk of encountering a pair of knots so awkward that we will not be able to decide whether the knots are equivalent. To date, fortunately, this has not happened. In principle, there *does* exist an algorithm, due to Haken and Hemion [Hak, Hem], for deciding whether or not two given knots are equivalent. However, to the best of our knowledge, this algorithm has not been implemented except in a few isolated cases.

[13]Thistlethwaite has different techniques for demonstrating primality, but *SnapPea*'s method is much more efficient.

Table 2. Classes of symmetry group.

Class	Symmetries contained in group	Symmetry type of knot
c	Type 0 symmetries only	Chiral, noninvertible
i	Type 0 and 1 symmetries only	Chiral, invertible
+	Type 0 and 2 symmetries only	+ Amphicheiral, noninvertible
−	Type 0 and 3 symmetries only	− Amphicheiral, noninvertible
a	Type 0, 1, 2, 3 symmetries	Fully amphicheiral, invertible

passing through the center of the diagram, the knot is mapped to itself with its orientation reversed; therefore, this rotation is a symmetry of type 1. We can also achieve a symmetry of type 0 for this knot, by rotating through half a turn about an axis normal to the projection plane.

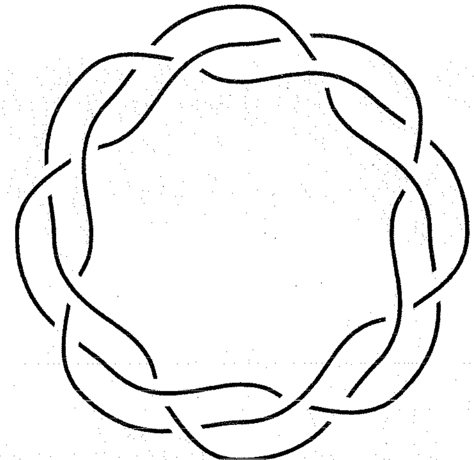
We can now classify symmetry groups according to which types of symmetry they contain. We observe that if a symmetry group contains two of the three types 1, 2, and 3, then it must contain the third one as well: simply take the product. Every symmetry group contains the identity, which is of type 0. Therefore, there are exactly five classes of symmetry groups, which we present in Table 2.

A handy feature of *SnapPea* is that it is able to compute the symmetry group of a hyperbolic knot and also the type of each symmetry within the group. It accomplishes this by finding all the ways of mapping the canonical triangulation onto itself. An obvious question is the following: which groups can be symmetry groups of knots? In the case of hyperbolic knots, it turns out that the group has to be finite and, moreover, has to be cyclic or dihedral [Ril, KS]. For nonhyperbolic knots, the characterization of symmetry groups is slightly more complicated.

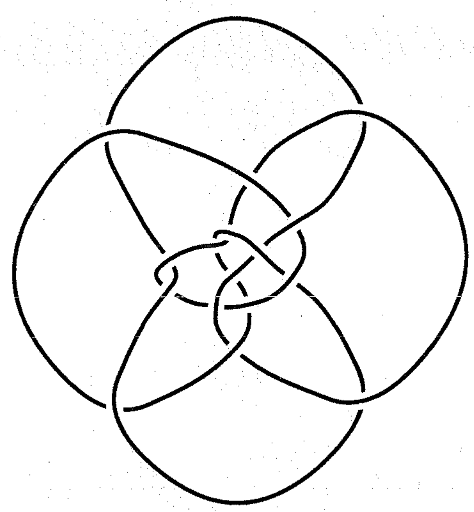
It so happens that all knots with crossing number ≤ 8 are either amphicheiral or invertible; however, the hordes of "random" knots soon begin to take over, and as Table A2A shows, at the level of 15 or 16 crossings almost all knots are completely asymmetric. We round off this section with a look at two 16-crossing exceptions to this trend toward chaos: (i) the "most symmetric" hyperbolic knot in the table, with symmetry group D_{16} and (ii) the only hyperbolic knot in the table with symmetry group D_9 (Fig. 7). Although the diagram of the D_{16} knot in Figure 7 displays its entire symmetry group in an obvious way, the diagram of the D_9 knot looks most unremarkable.

The fact that the symmetry group of the D_{16} knot is so visible is no coincidence; it is proved in [MT] that any symmetry of a prime alternating link must be visible, up to flypes, in any alternating diagram of the link.[14] Even if the alternating diagram admits nontrivial flypes, it is a relatively simple matter to determine the symmetry group of the link by inspection of the diagram, although this procedure does require a modest amount of practice. But the minimal diagram of the D_{16} knot does not admit any nontrivial flype, so the symmetries of this knot must be immediately apparent, in the sense that they correspond to symmetries of the diagram.

The symmetries of the D_9 knot, by way of contrast, are not remotely visible in the 16-crossing diagram of Figure



The D_{16} knot



The D_9 knot

Figure 7

7. However, one way of revealing the symmetries is to construct the knot by cyclically pasting together nine identical pieces, as in Figure 8. Each piece, shown dark gray in the picture, is a "bundle" of five arcs, and two adjacent pieces differ by a "twist" of a third of a turn.

The Nonhyperbolic Knots

As already mentioned, there are 32 nonhyperbolic knots with 16 or fewer crossings, of which 12 are torus knots and the remaining 20 are satellites of the trefoil.

A torus knot is a simple closed curve sitting on a standardly embedded torus. When we speak of a "(p,q)-torus knot," we mean a simple closed curve on the torus which wraps around p times meridianally and q times longitudinally. The integers p and q are necessarily relatively prime: otherwise, we would have a torus *link*. In Figure 2, the second knot in the first row is a (3, 2)-torus knot, the last knot

[14]A similar result was obtained by Bonahon and Siebenmann for "algebraic" links [BS].

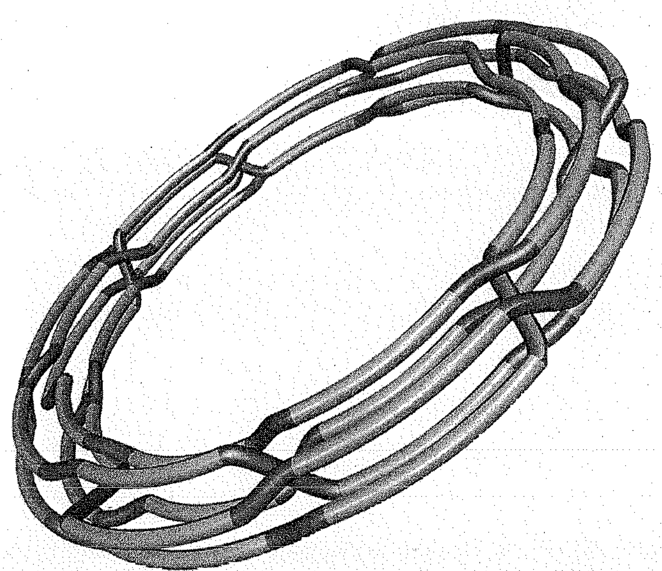


Figure 8. A symmetric representation of the D_9 knot. It was rendered using Larry Gritz's Blue Moon Rendering Tools.

in the first row is a (5, 2)-torus knot, and the last knot in the third row is a (7, 2)-torus knot. From this, the reader can easily produce analogous diagrams of (p, q)-torus knots for other values of p and q and will observe that each diagram thus constructed has $p(q-1)$ crossings. It is not hard to show that for a given p and q, a (p, q)-torus knot is isotopic to a (q, p)-torus knot and that torus knots are classified up to unoriented equivalence by the unordered pair of positive integers p and q. Henceforth, without fear of ambiguity, we shall speak of *the* (p, q)-torus knot.

Because interchanging p and q does not alter the knot type, it follows that the (p, q)-torus knot has two "natural" diagrams, one with $p(q-1)$ crossings and the other with $q(p-1)$ crossings. It is proved in [Wil, MP] that the crossing number is the smaller of these two numbers. From this result on crossing number and the fact that p and q must be relatively prime, the values of (p, q) giving a torus knot with 16 or fewer crossings are (3, 2), (5, 2), (7, 2), (9, 2), (11, 2), (13, 2), (15, 2), (4, 3), (5, 3), (7, 3), (8, 3), and (5, 4).

It is well known [Sch] that torus knots are all chiral and invertible and that each has symmetry group D_1. Therefore, our discussion of torus knots is complete, and we turn to satellite knots.

If a knot K is placed inside a solid torus V, and V is itself knotted in $\mathbb{R}^3$, then K is called a *satellite knot* (certain mild restrictions must be imposed on the way in which K is placed in V, to avoid trivial cases). The core C of V (i.e., the knot traveling around the center of V) is called a *companion* of K, and we say that K is a *satellite of* C. Figure 9 illustrates a satellite knot[15] of the trefoil (i.e., the knot is sitting inside a solid torus which is a thickened trefoil).

Let us suppose that the companion knot has crossing number k and that the satellite wraps (or ravels) m times longitudinally around the solid torus; for example, in Figure 9, we have $k = 3$ and $m = 2$. There is an obvious diagram of the satellite where at each crossing of the companion we see an $m \times m$ "grid" of crossings of the satellite. Such a diagram has at least km^2 crossings, and it is an unproven "factoid" of knot theory that the satellite cannot be projected with fewer than km^2 crossings.

Because the trefoil has 3 crossings, any satellite of it appearing in our table should have wrapping number 2 (a higher wrapping number should entail at least 27 crossings). Moreover, we should not expect any satellite of the figure-eight knot to have fewer than $(4)(2^2) + 1 = 17$ crossings, where $+1$ refers to an additional crossing needed to produce a knot rather than a link of two components. Our table does not contain any counterexample to the conjecture: the satellites through 16 crossings all wrap twice around the trefoil, as in Figure 9.

Because the trefoil is chiral, it follows from a standard result of knot theory that these 20 satellites are also chiral. Moreover, it is not hard to see that they are all invertible: if we rotate through half a turn about a "north–south" axis in the projection plane passing through the substituent tangle, the tangle is flipped over and the rest of the satellite knot is mapped onto itself with reversed orientation. But each of the 10 tangles is 2-bridged, and it is well known that 2-bridged tangles are invariant (up to isotopy) under such a rotation.

The symmetry groups of the satellite knots were computed: the four cable knots[16] (i.e., the knots with substitutent tangles , , ,) have symmetry groups D_3, D_5, D_7 and D_9, respectively, whereas each of the remaining 16 satellite knots has infinite dihedral symmetry group. In general, the symmetry group of a satellite knot need not be cyclic or dihedral.

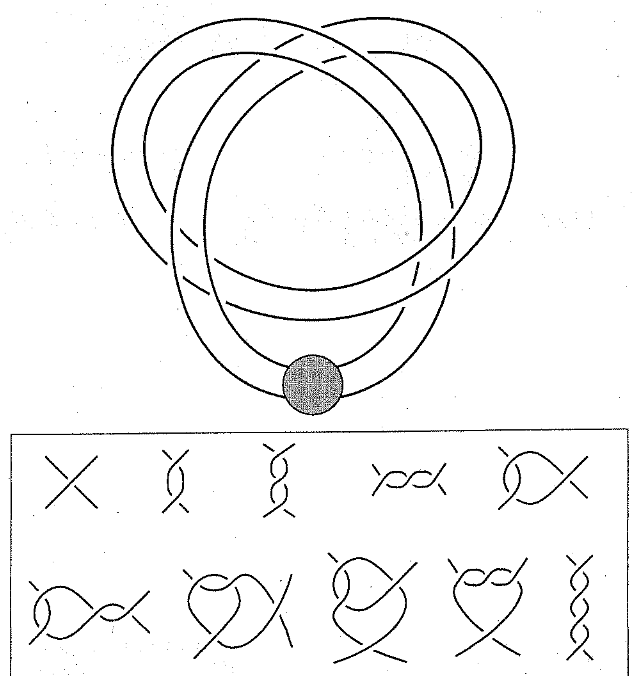


Figure 9. Each satellite knot with ≤16 crossings is obtained by substituting one of these tangles, or its reflection, into the shaded disk.

[15]More precisely, it is a *template:* it becomes a knot when 1 of the 10 tangles in the box underneath is substituted for the shaded disk.

[16]A satellite knot K is a *cable knot* if it lies on the boundary of the companion solid torus V [BZ].

Appendix I: Summary Data

Table A1

# Crossings	# Knots	# Torus	# Sat.	c	+	−	i	a
0	1							1
3a	1	1					1	
4a	1							1
5a	2	1					2	
6a	3						2	1
7a	7	1					7	
8a	18					1	13	4
8n	3	1					3	
9a	41	1		2			39	
9n	8						8	
10a	123			21		6	89	7
10n	42	1		6			36	
11a	367	1		123			244	
11n	185			64			121	
12a	1,288			637	1	37	597	16
12n	888			466		3	418	1
13a	4,878	1		3,261			1,617	
13n	5,110		2	3,658			1,452	
14a	19,536			15,229	5	183	4,084	35
14n	27,436	1	2	22,656	1	44	4,729	6
15a	85,263	1		74,223			11,040	
15n	168,030	1	6	152,357		1	15,672	
16a	379,799			350,260	40	927	28,490	82
16n	1,008,906	1	10	958,189	25	434	50,227	31

Table A2A. Distribution of symmetry groups of hyperbolic knots.

	Z_1	Z_2		Z_3	Z_4	D_1		D_2			D_3	D_4		D_5	D_6		D_7	D_8		D_9	D_{10}		D_{14}	D_{16}
# Crossings	c	c	+	c	+	−	i	−	i	a	i	i	a	i	i	a	i	i	a	i	i	a	a	a
4a											1													
5a									1															
6a									2			1												
7a									4			2												
8a						1	2		11			3							1					
8n							1		1															
9a	2						9		22		1	4			2									
9n							4		1		2				1									
10a	19	2				6	35		53		1		5		1							1		
10n	5	1					25		9															
11a	116	7					120		112			1												
11n	57	7					97		22		2	10												
12a	602	35	1			32	350	5	242	1	2	1	13		2	2								
12n	445	21				2	344	1	60		4	4			1	3	1	2						
13a	3,154	107					1,080		511			21			2			2						
13n	3,555	103					1,311		136		1			1	1									
14a	14,851	378	4		1	179	2,998	4	1,084	11			23		2								1	
14n	22,326	330			1	42	4,356	2	358	4	4	6	2	2										
15a	73,185	1,037		1			8,695		2,265		16	51		1	7			2			2			
15n	151,126	1,231				1	14,821		800		33	2		2	5						2			
16a	347,111	3,149	33		7	891	23,725	36	4,754	25	2	3	54		5				2		1			1
16n	954,381	3,808	21		4	423	48,224	11	1,941	24	13	27	5		4	1	1	5	1	1				

Table A2B. Distribution of symmetry groups of satellite knots.

	D_3	D_5	D_7	D_9	D_∞
# Crossings	**i**	**i**	**i**	**i**	**i**
13n		1	1		
14n					2
15n	1			1	4
16n					10

Appendix II: Tables of Hyperbolic Knots with Selected Symmetries

For compactness, we have adopted the following alphabetic encoding of the DT notation.

a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
−2	−4	−6	−8	−10	−12	−14	−16	−18	−20	−22	−24	−26	−28	−30	−32

Knots with +amphicheiral symmetry

beifhakdlcgj Z2
bfjgiaeldncmhk Z2
bgihkmjacnldfe Z4
cflnhkbjmegadi Z2
dfgljabkmnechi Z2
dfgljbakmnecih Z2
bdganhmcjloifpek Z2
bdganpmcjloifhek Z2
begnajpdlofmichk Z2
belfiaojdhpmckgn Z2
bfglhkacmnepdoij Z2
bfgljapdmoecnihk Z2
bfhlnjaicgpdkoem Z4
bfkhjgaendpcoiml Z2
bfkhojalndpgcime Z2
bfkohjaendgpciml Z2
bflghjadneopcimk Z2
bflhgjadnepocimk Z2

bgempkajnlfhdoic Z2
bgihlnkajcpmdofe Z4
bglmhjkaoefpcdin Z2
bglmhjkaofepdcin Z2
bhijkmpoanedflcg Z4
bhjilknapcomegfd Z4
bhjilonagcpmekfd Z4
cdganhmbkloifpej Z2
cdganpmbkloifhej Z2
cegnahlbkmofipdj Z2
cegnhblakmofpjdi Z2
cfglhkbanoepdjim Z2
cfmjohlaknebgpdi Z2
cgenpblakomfhjdi Z2
cgfjiplmkonbahde Z2
cgflhkbaonepdjim Z2
cglnipbmkodfahje Z2
cgmnhibdkoefpajl Z2

cgmpniblkoehfajd Z2
cgnmhidbkofepalj Z2
chjmglnakpbeodfi Z4
chlnoibmkpdfgaje Z2
chmnoilbkpefgadj Z2
chmonilbkpegfadj Z2
clfgibpmkdnoajhe Z2
clfoipbmkdngahje Z2
dehnaojclmpfigbk Z2
fhkmjloanpcebdgi Z4
bfhgIlacKMDnJe Z4
bdFJaILmCEnPOkhG Z4
bdGaNHMCjlOiFPEK Z2
bdGaNPMCjlOiFHEK Z2
bdGaNhCmJLoIfPkE Z2
bdGanHCMJLoIFpke Z2
bdJFaILOCEnHPkmG Z4
bdgaNhcmJLOIfPKE Z2

bdganHcMJLOIFpKe Z2
beGNaJPDlOFmiCHK Z2
begnaJdLPFIMHocK Z2
bfKHOJaLnDPGCimE Z2
bfLgjhadNeOCPIMK Z2
bfhgKnaicLoDpemj Z4
bfhlgJadNPEOcIMK Z2
bgIhkNajoPledFMC Z4
bhEMGkCaOlfnDjPI Z2
bheMgkcaOLfNDJPI Z2
cEnoHBIDKmFGpjal Z2
ceghNblaKMOPfJDI Z2
cegnaLKbFOPHIJdM Z2
cehNgblaKMPfOJDI Z2
ceoHnbIDKMGpFJal Z2
cfmoHIbDKNEGpaJl Z2
cgnmHIbDKOFEpaJl Z2
cnoHabMDKFGpIJel Z2

Knots with Z_3, Z_4 symmetry

dfholikmebnacjg c Z3
bgihkmjacnldfe + Z4
bfhlnjaicgpdkoem + Z4
bgihlnkajcpmdofe + Z4

bhijkmpoanedflcg + Z4
bhjilknapcomegfd + Z4
bhjilonagcpmekfd + Z4
chjmglnakpbeodfi + Z4

fhkmjloanpcebdgi + Z4
bfhgIlacKMDnJe + Z4
bdFJaILmCEnPOkhG + Z4
bdJFaILOCEnHPkmG + Z4

bfhgKnaicLoDpemj + Z4
bgIhkNajoPledFMC + Z4

Knots with D_5 symmetry

cemfhaikdlngobj i
deflijcKabGH i

beKFhaJmdLCGI i
beKGaHJDmFLCni i

bflNhaDKeMGicJ i
cEMfHAiKDlNGoBJ i

defLhmjkaCnoIbg i

Knots with D_6 symmetry

chgfbaied i	bfgilaejchmdk i	fijlmknocbaedhg i	beGFhaCid i	cihLKJnmbaEDogf i
dfhgibace i	fhijklmacbedg i	fjimlkoncbaedhg i	bdJFaIKCEHLG i	eImKLjNcABoDhFG i
befgiahcjd i	bghijkmacdlenf i	gijklmnoabdcfeh i	cFEhgJIlkBAd i	ehGkinmLaOdcBfJ i
eghijkbadcf i	bghiklanjdcmfe i	bfhjngaelkcipodm i	cJElgBIdkFAh i	beNGaHJDlFmpiOCk i
bfghikacjdle i	bfjhmgaokciendl i	bhijklmoacdenfpg i	ceFhGjIKlBaD a	bfKhIaLoNGPEJCdM a
bfhgjalicked i	bghiknafcldjoem i	bhijkmnapcledogf i	cEfjhkmdlgaBi i	cENfhAJdKMGoIpBl i
cfehgjilkbad a	eimkljncabodhfg i	chijklnobamedpgf i	beiHLajNOgfMDKC i	cGKhJNLAmOFBDpEI i
cjelgbidkfah a	eimlkjncbaodhgf i	cihjlkonbamedpgf i	chiKLJmnbaEDogf i	cfglijkPmnbaoedH i

Knots with $D_7 - D_{16}$ symmetry

efgIOklmCNpabhDj i D7	dkfmhojalcnepgbi a D8	dKFmhOJalCNepGBi i D8	bgIHjlnaCkdmeof i D10
cdefghab a D8	bfHGikaCjdle i D8	deigkNjPlmaocFbH a D8	bhJIKMLnaCEDFog i D10
bfghjlaickdme i D8	bgIHJLKaCEDF i D8	dJilKnAcOMFpgBeH i D9	efghijklmnabcd a D14
fhijlkmbadceg i D8	bfHKNapICGLDJOEM i D8	defghijabc a D10	fghijklmnopabcde a D16
bghijlnackdmeof i D8	bgihkNapjcledOFM i D8	dinglbjoemchafk i D10	
gijklmnobadcfeh i D8	cEonhBIdKmGFPjaL i D8	gikjlnmobacedfh i D10	
cnepgbidkfmhojal a D8	cNEpgBIdkFMhoJAl i D8	bghikmoajcldnepf i D10	

Appendix III: *Knotscape* and *SnapPea*

Knotscape is primarily a graphical interface to the knot tables. It is currently still in the development phase, but it already allows the user to browse through the knot tables and locate user-supplied knots in the tables. It will display a picture of the knot currently selected and will compute polynomial invariants and a few other invariants. The graphical part of the program is written in Tcl7.4/Tk4.0, and the computational modules are written in C. It has been tested on Linux systems and on Sun systems. The program is available for download from http://www.math.utk.edu/~morwen.

SnapPea is an interactive computer program for creating and studying hyperbolic 3-manifolds. At present, the most full-featured version runs on a Macintosh and is available for free from www.geom.umn.edu. (For current information about other platforms, please contact weeks@geom.umn.edu.) *SnapPea* works with arbitrary closed and cusped hyperbolic 3-manifolds. Initially, the user specifies a manifold by drawing a knot or link and asking *SnapPea* to construct its complement, or by selecting a manifold from *SnapPea*'s built-in databases, or by some other method. Thereafter, the user can create new manifolds from old ones by taking finite-sheeted covers (or branched covers) by drilling out closed geodesics to create new cusps, or by doing Dehn fillings to seal off old cusps. For all manifolds, *SnapPea* computes a wide variety of numerical, algebraic, and graphical invariants.

REFERENCES

[AB] J.W. Alexander and G.B. Briggs, On types of knotted curves, *Ann. Math.* 28 (1927), 562–586.

[Arn] B. Arnold, M. Au, C. Candy, K. Erdener, J. Fan, R. Flynn, J. Hoste, R.J. Muir, and D. Wu, Tabulating alternating knots through 14 crossings, *J. Knot Theory Ramifications* 3(4) (1994), 433–437.

[BS] F. Bonahon and L. Siebenmann, The classification of algebraic links, unpublished manuscript.

[BZ] G. Burde and H. Zieschang, Knots, New York: de Gruyter (1985).

[Cau] A. Caudron, Classification des noeuds et des enlacements, Prepublication Math. d'Orsay, Orsay, France: Université Paris-Sud (1980).

[Con] J.H. Conway, An enumeration of knots and links, *Computational Problems in Abstract Algebra* (Leech, ed.), New York: Pergamon Press (1970), 329–358.

[Deh] M. Dehn, Die beiden Kleeblattschlingen, *Math. Ann.* 75 (1914), 402–413.

[DH] H. Doll and J. Hoste, A tabulation of oriented links, *Math. Computat.* 57(196) (1991), 747–761.

[DT] C.H. Dowker and M.B. Thistlethwaite, Classification of knot projections, *Topol. Appl.* 16 (1983), 19–31.

[EP] D.B.A. Epstein and R.C. Penner, Euclidean decompositions of noncompact hyperbolic manifolds, *J. Diff. Geom.* 27 (1988), 67–80.

[ES] C. Ernst and D.W. Sumners, The growth of the number of prime knots, *Math. Proc. Camb. Phil. Soc.* 102 (1987), 303–315.

[GL] C. Gordon and J. Luecke, Knots are determined by their complements, *J. Am. Math. Soc.* 2 (1989), 371–415.

AUTHORS

JIM HOSTE
Pitzer College
Claremont, CA 91711, USA
jhoste@pitzer.claremont.edu

Jim Hoste received his Ph.D. from the University of Utah in 1982, spent a year at the Courant Institute of Mathematical Sciences as a National Science Foundation Postdoctoral Fellow, and is now a professor at Pitzer College, one of the Claremont Colleges. He works primarily in knot theory and is best known as the "H" in the HOMFLY polynomial, an invariant of knots encompassing both the Jones polynomial and the Alexander polynomial. When not tied up at the office, he enjoys mountain biking, climbing, skiing, music, and the company of his wife and two sons.

MORWEN THISTLETHWAITE
University of Tennessee
Knoxville, TN 37996, USA
morwen@math.utk.edu

Morwen Thistlethwaite is a professor of mathematics at the University of Tennessee, specializing in knot theory. He is a native of London, England, and came to the United States in 1987. He received his mathematical training at the universities of Cambridge, London, and Manchester. At one time he seriously considered a career as a pianist, and he has given many solo concerts in England.

JEFF WEEKS
Canton, NY 13617, USA
weeks@geom.umn.edu

Jeff Weeks has an A.B. from Dartmouth College and a Ph.D. from Princeton University, both in mathematics. Now an independent consultant, he splits his time between research, education, and his family. Though primarily a topologist and geometer, he has recently fallen in with a gang of cosmologists that hopes to determine the global topology of the universe from the cosmic microwave background radiation.

[Hak] W. Haken, Theorie der Normalflächen, *Acta Math.* 105 (1961), 245–375.

[Has1] M.G. Haseman, On knots, with a census of the amphicheirals with twelve crossings, *Trans. Roy. Soc. Edinburgh* 52 (1917), 235–255.

[Has2] M.G. Haseman, Amphicheiral knots, *Trans. Roy. Soc. Edinburgh* 52 (1918), 597–602.

[Hem] G. Hemion, On the classification of homeomorphisms of 2-manifolds and the classification of 3-manifolds, *Acta Math.* 142 (1979), 123–155.

[Kau] L.H. Kauffman, State Models and the Jones Polynomial, *Topology* 26 (1987), 395–407.

[Kir1] T.P. Kirkman, The enumeration, description and construction of knots of fewer than ten crossings, *Trans. Roy. Soc. Edinburgh* 32 (1885), 281–309.

[Kir2] T.P. Kirkman, The 634 unifilar knots of ten crossings enumerated and defined, *Trans. Roy. Soc. Edinburgh* 32 (1885), 483–506.

[KS] K. Kodama and M. Sakuma, Symmetry groups of prime knots up to 10 crossings, in *Knots 90, Proceedings of the International Conference on Knot Theory and Related Topics, Osaka, Japan, 1990* (A. Kawauchi, ed.), Berlin: de Gruyter (1992), 323–340.

[Lis] J.B. Listing, Vorstudien zur Topologie, Göttingen Studien, University of Göttingen, Germany (1848).

[Lit1] C.N. Little, On knots, with a census of order ten, *Trans. Connecticut Acad. Sci.* 18 (1885), 374–378.

[Lit2] C.N. Little, Non alternate $\pm$ knots of orders eight and nine, *Trans. Royal Soc. Edinburgh* 35 (1889), 663–664.

[Lit3] C.N. Little, Alternate $\pm$ knots of order 11, *Trans. Roy. Soc. Edinburgh* 36 (1890), 253–255.

[Lit4] C.N. Little, Non-alternate $\pm$ knots, *Trans. Royal Soc. Edinburgh* 39 (1900), 771–778.

[Lit5] C.N. Little, Knots, with a census for order ten, Ph.D. Thesis, Yale University (1885).

[Men] W. Menasco, Closed incompressible surfaces in alternating knot and link complements, *Topology* 23(1) (1984), 37–44.

[MP] K. Murasugi and J. Przytycki, The skein polynomial of a planar star product of two links, *Math. Proc. Camb. Phil. Soc.* 106(2) (1989), 273–276.

[MT] W. Menasco and M.B. Thistlethwaite, The classification of alternating links, *Ann. Math.* 138 (1993), 113–171.

[Mur1] K. Murasugi, The Jones polynomial and classical conjectures in knot theory, *Topology* 26 (1987), 187–194.

[Mur2] K. Murasugi, Jones polynomials and classical conjectures in knot theory II, *Math. Proc. Camb. Phil. Soc.* 102 (1987), 317–318.

[Per1] K. Perko, On the classification of knots, *Proc. Am. Math. Soc.* 46 (1974), 262–266.

[Per2] K. Perko, Invariants of 11-crossing knots, Prepublications Math. d'Orsay (1980)

[Per3] K. Perko, Primality of certain knots, *Topology Proc* 7 (1982), 109–118.

[Pra] G. Prasad, Strong rigidity of Q-rank 1 lattices, *Invent. Math.* 21 (1973), 255–286.

[Rei] K. Reidemeister, Knotentheorie, Berlin: Springer-Verlag, 1932.

[Ril] R. Riley, An elliptical path from parabolic representations to hyperbolic structures, in *Topology of Low-Dimensional Manifolds, Proceedings, Sussex 1977.* (R. Fenn, ed.), Springer Lecture Notes in Math. vol. 722, New York: Springer-Verlag (1979), 99–133.

[SW] M. Sakuma and J. Weeks, The generalized tilt formula, *Geometriae Dedicata* 55 (1995), 115–123.

[Sch] O. Schreier, Über die Gruppen $A^aB^b = 1$, *Abh. Math. Sem. Univ. Hamburg* 3 (1924), 167–169.

[ST] C. Sundberg and M. Thistlethwaite, The rate of growth of the number of prime alternating links and tangles, *Pacific J. Math.* 182 no.2 (1998), 329–358.

[Tai] P.G. Tait, On knots I, II, III, *Scientific Papers, Vol. I*, Cambridge: Cambridge University Press (1898), 273–347.

[Thi1] M.B. Thistlethwaite, Knot tabulations and related topics, *Aspects of Topology in Memory of Hugh Dowker 1912–1982* (James and Kronheimer, eds.), Cambridge, England: Cambridge Press, London Math. Soc. Lecture Note Series 93, 1–76.

[Thi2] M.B. Thistlethwaite, A spanning tree expansion of the Jones polynomial, *Topology* 26(3) (1987), 297–309.

[Thi3] M.B. Thistlethwaite, Kauffman's polynomial and alternating links, *Topology* 27(3) (1988), 311–318.

[Thi4] M.B. Thistlethwaite, On the structure and paucity of alternating tangles and links, preprint.

[Tro] H.F. Trotter, Noninvertible knots exist, *Topology* 2 (1964), 275–280.

[Wks] J. Weeks, Convex hulls and isometries of cusped hyperbolic 3-manifolds, *Topol. Appl.* 52 (1993), 127–149.

[Wel] D.J.A. Welsh, On the number of knots and links, *Colloq Math. Soc. J. Bolyai* 60 (1991), 713–718.

[Wil] R.F. Williams, The braid index of an algebraic link, *Braids (Santa Cruz, CA, 1986)*, Contemporary Mathematics Series Vol. 78, Providence, RI: American Mathematical Society (1988).