

The Fundamental Group of a Link

Notes for Math 148

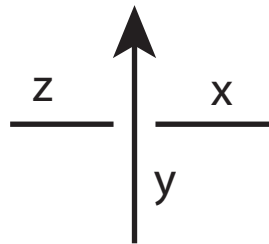
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1 The Wirtinger Presentation

Associated to every knot or link in \mathbb{R}^3 (or in S^3) we can associate a group, called the *fundamental group* of the complement of the link, or more simply, the fundamental group of the link. A simple-minded way to approach this is to extend the idea of labeling a link diagram by group elements. Given a group G and an oriented link diagram D , we say that D can be labeled with G if we can label every arc of the diagram with an element from G such that

1. The set of group elements used to label the diagram generate the group, and
2. At each crossing, the three group elements that appear there satisfy the relation $xy = yz$ where the crossing and labels appear as shown.



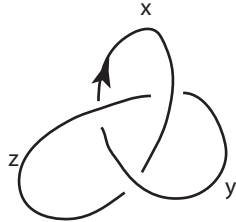
Note that the relation at each crossing depends on the orientation of the over-crossing strand, but not of the under-crossing strand.

It is not hard to show that if an oriented diagram can be labeled with a group G , then changing the diagram by a Reidemeister move gives a diagram

that can also be labeled with G . Thus being labeled with G is really a property of the link, independent of the choice of diagram.

Instead of starting with a given oriented diagram and a given group and seeing if it is possible to label the diagram with the group, we can start with the oriented diagram and *define* a group G with which it is possible to label the diagram! We simply label each arc of D with a different symbol and then define G to be the group generated by these symbols subject to the relations determined at each crossing. Such a definition of a group is called a *group presentation*.

Example Consider the diagram of the right-handed trefoil shown below.



The group is defined by three generators and three relations.

$$G = \langle x, y, z \mid yz = zx, zx = xy, xy = yz \rangle .$$

The elements of this group are all words in the letters $\{x, y, x^{-1}, y^{-1}, z^{-1}\}$. Thus $xyxzzzx^{-1}$ and $zyy^{-1}z^{-1}$ are elements in the group. The empty word is also an element, which we write as 1. Two elements are multiplied by concatenating the two words. When we do this we might create an instance of gg^{-1} where g is one of the generators or its inverse. When this happens we “cancel” the pair of letters. Thus xy^{-1} times yz gives $xy^{-1}yz = xz$ for example. It follows that 1 is the identity element of the group. In addition, the relations also allow us to rewrite words in the group. Since in this group $yz = zx$, if we ever see a word that contains yz then we may replace those two letters with zx . Thus for example, $y^2zx^{-1} = yzxx^{-1} = yz$.

The presentation of the group we obtain in this way from an oriented link diagram is called the *Wirtinger* presentation and the generators are called *Wirtinger* generators.

Different presentations are possible of the same group. This is because we can change a presentation by introducing or deleting redundant generators or by introducing or deleting a relation that is a consequence of all the other relations. We illustrate this with the above presentation. Notice that we can solve the last relation for z obtaining $z = y^{-1}xy$. This means that we do

not need z as a generator since we can obtain z from x and y . We therefore eliminate z from the list of generators and moreover, wherever z appears in a relation, we replace it with $y^{-1}xy$. We obtain

$$\langle x, y \mid yy^{-1}xy = y^{-1}xyx, y^{-1}xyx = xy \rangle .$$

After canceling the yy^{-1} in the first relation, we see that the two relations are the same. Thus we can remove one of them and obtain

$$\langle x, y \mid y^{-1}xyx = xy \rangle$$

which we rewrite as

$$\langle x, y \mid xyx = yxy \rangle .$$

It turns out that given any oriented link diagram D , any one of the relations in the Wirtinger presentation will always be a consequence of the other relations and so may be eliminated. Thus every knot diagram, or connected link diagram, produces a group with one more generator than relation. If the diagram is not connected, then the number of generators minus the number of relations can be larger than 1. An important theorem of Schreier states that any group having a presentation with more generators than relations is infinite. Thus the group associated to any link diagram is always infinite.

If we change the diagram of a link with a Reidemeister move, then one can prove that the Wirtinger presentation changes by either introducing or deleting a redundant generator, or by introducing or deleting relations that are consequences of the other relations. Hence we obtain the same group no matter what diagram we use to represent a given link. Thus the group associated to an oriented link diagram is actually an invariant of the underlying oriented link.

Example Let K be the unknot. Using the diagram with no crossings, the Wirtinger presentation consists of 1 generator and no relations. This defines the infinite cyclic group \mathbb{Z} . It is an important result that the unknot is the ONLY knot whose fundamental group is infinite cyclic.

An important observation about the Wirtinger presentation of the group of a link is that all the Wirtinger generators that label a given component are conjugate. Thus in the case of a knot, all the Wirtinger generators are conjugate to each other.

2 A More Topological Point of View

In the previous section we saw how to associate a group to an oriented link in a purely combinatorial way; we assigned a group presentation to a diagram and then observed that presentations derived from different diagrams of the same link presented the same group. However, we can define the group in an entirely different way.

Pick a point P in the complement of the link and call P the basepoint. Now consider a loop based at P and lying in the complement of the link. This is a curve $\sigma(t) = (x(t), y(t), z(t))$ with $0 \leq t \leq 1$ where x, y and z are continuous functions of t , $\sigma(t)$ is in the complement of the link for all t , and $\sigma(0) = \sigma(1) = P$. We can define a multiplication of two loops α and β as follows

$$(\alpha \cdot \beta)(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2 \\ \beta(2t - 1) & 1/2 \leq t \leq 1 \end{cases}$$

This corresponds to first traversing the loop α and then traversing the loop β .

One might hope that this turns the set of loops into a group, but it does not. The operation is not associative, for example. We can create a group using loops if we use as objects, not the loops themselves, but instead equivalence classes of loops where two loops are considered equivalent if one can be deformed into the other in a continuous way. More precisely, we say that loops α and β are *homotopic* if there exists a function $f(s, t) = (x(s, t), y(s, t), z(s, t))$ with $0 \leq s \leq 1$, $0 \leq t \leq 1$, and with x, y, z continuous functions of s and t such that:

1. $f(s, t)$ is in the complement of the link for all s and t ,
2. $f(0, t) = \alpha(t)$,
3. $f(1, t) = \beta(t)$,
4. $f(s, 0) = f(s, 1) = P$ for all s .

It is now possible to prove that multiplying two loops provides a well-defined operation on the set of homotopy classes of loops and moreover turns the set of homotopy classes into a group. The identity element of the group is the constant loop $id(t) = P$ for all t , and the inverse of the loop $\alpha(t)$ is the loop $\alpha(1-t)$ obtained from α by traversing it in reverse. This group is called

the fundamental group of the complement of the knot and turns out to be the same group given by the Wirtinger presentation. (Note that historically, the fundamental group was first described in this way and then Wirtinger showed how to describe it with generators and relators given a diagram of the link.)

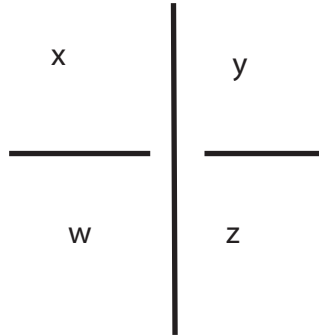
The correspondence between the two descriptions of the fundamental group is as follows. Suppose x is a Wirtinger generator. Then x corresponds to the loop that leaved the base point, goes down through the projection plane just to the right of the arc labeled x , goes under the arc, passes back up through the projection plane just to the left of the arc, and then returns to the base point. Here “right” and “left” of the arc are determined by the orientation of the arc, so that the loop has linking number $+1$ with the knot.

Once we know that these two different descriptions of the fundamental group of a link describe the same group, it follows that the orientation of the link is irrelevant—reorienting the link does not change its complementary space, the only thing upon which the group depends. It also follows that it really only depends on the link, not any particular diagram of the link, again because it only depends on the complement of the link.

3 The Dehn Presentation of the Fundamental Group

Starting with an oriented link diagram D , Dehn gave a presentation of the fundamental group of the complement of D with one generator for each *region* of the diagram. (If we replace the crossing of D with actual double points, then the regions of the diagram are simply the complementary regions of the link projection in the projection plane.) Imagine the diagram as lying nearly in the xy -plane in \mathbb{R}^3 and consider a base point high above the plane of the diagram. Given a region R of the diagram, define a loop α_R as follows. Leave the base point and pass down through the region R . Then travel under the plane of the diagram until the unbounded region is reached and return to the base point up through the unbounded region. Note that the loop α_R where R is the unbounded region of the diagram is the trivial loop since it goes down and then back up through the same region. It is not hard to prove that the loops corresponding to the regions generate the fundamental group. At each crossing we obtain a relation. Consider a

crossing and the four regions that surround the crossing as shown below.



Each region represents a loop in the fundamental group. The loop xy^{-1} goes down through the region labeled x and then up through the unbounded region, then back down through the unbounded region and up through the region labeled y . This is equivalent to going down through x and up through y . But this loop can be slid down in the figure to be the same as the loop wz^{-1} . Thus we obtain the relation $xy^{-1} = wz^{-1}$. Writing down one generator for each region and one relation for each crossing, together with the relation $\alpha_R = 1$ where R is the unbounded region, gives the Dehn presentation of the fundamental group.

As with the Wirtinger presentation, it turns out that any single relation is a consequence of all the others. But unlike the Wirtinger presentation, the generators are not all conjugate to each other.