Factoring Polynomials

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1 Factoring

In the previous section, we discussed how to determine the product of two or more terms. Consider, for instance, the equations below, where we multiply the expressions on the left to produce the expressions on the right:

$$x(x+2) = x^2 + 2x$$

$$(w-7)(w+7) = w^2 - 49$$

$$2(z-3)(z+2)(z^2+5) = 2z^4 - 2z^3 - 2z^2 - 10z - 60$$

In this section, we discuss how to "undo" this process. There are times when the expression on the right is much simpler to work with. However, the left side can provide valuable information as well. We want to be able to take an expression in either form, and convert it to the other. In this section, we discuss how to "undo" the process of multiplication. This is called *factoring*. We will see later that factoring is often used to answer the very important question: "For what values of a variable is an expression equal to zero?" So we see that there are now two points of view from which to look at an equation like:

$$(r-6)(r+8) = r^2 + 2r - 48$$

These are:

1. The product of $(r-6)$ and $(r+8)$ may be expressed as $r^2 + 2r - 48$
2. The factors of the expression $r^2 + 2r - 48$ are $(r-6)$ and $(r+8)$.

In the previous section, we saw that multiplication of algebraic terms is very straightforward, in fact what we might call "cook book". Factoring isn't quite as simple. There are several rules and formulas to recall (or look up), and it is crucial to know when to use these. In this section, we will discuss these rules, and how to use them, providing several examples of each.

Below we will list several rules used to factor algebraic expressions. Most of these are factoring instructions, based on the form of the expression needing to be factored. In this section, we will show how to factor only polynomial expressions. Later, when we encounter more complicated types of expressions, we can easily show how the ideas presented here can be generalized.

I Factoring an Expression Using Simple Formulas

These first simple formulas involve the sums and differences of the same power of two terms. Their simplicity is due to the fact that the the power is low, either 2 or 3. Later we will generalize a few of these, using higher powered terms.

METHOD 1 **The Difference of Squares**

$$x^2 - a^2 = (x-a)(x+a)$$

Example 1

$$x^2 - 64 = (x-8)(x+8) \qquad v^2 - \tfrac{1}{196} = (v - \tfrac{1}{14})(v + \tfrac{1}{14})$$

$$w^2 - 5 = (w - \sqrt{5})(w + \sqrt{5}) \qquad h^2 - 2b^2 = (h - \sqrt{2}b)(h + \sqrt{2}b)$$

$$z^2 - r^6 = (z - r^3)(z + r^3) \qquad n^2 - 4^{\frac{2}{3}} = (n - 4^{\frac{1}{3}})(n + 4^{\frac{1}{3}})$$

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METHOD 2 **The Difference of Cubes**

$$x^3 - a^3 = (x-a)(x^2 + ax + a^2)$$

Example 2

$$\begin{aligned} x^3 - 64 &= (x-4)(x^2+4x+16) & v^3 - 1331 &= (v-11)(v^2+11v+121) \\ r^3 - 1 &= (r-1)(r^2+r+1) & t^3 - \pi^3 &= (t-\pi)(t^2+\pi t+\pi^2) \\ n^3 - 4 &= (n-4^{\frac{1}{3}})(n^2+4^{\frac{1}{3}}n+4^{\frac{2}{3}}) & b^3 - \tfrac{1}{729} &= (b-\tfrac{1}{9})(b^2+\tfrac{b}{9}+\tfrac{1}{81}) \end{aligned}$$

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> Method 3 **The Sum of Cubes**
>
> $$x^3 + a^3 = (x+a)(x^2 - ax + a^2)$$

Example 3

$r^3 + 1 = (r+1)(r^2 - r + 1)$

$x^3 + 64 = (x+4)(x^2 - 4x + 16)$

$t^3 + \frac{1}{\pi^3} = (t + \pi^{-1})(t^2 - \pi^{-1}t + \pi^{-2})$

$27L^3 + (2\pi)^3 = (3L + 2\pi)(9L^2 - 6L\pi + 4\pi^2)$

$z^3 + 1728 = (z+12)(z^2 - 12z + 144)$

$y^3 + 17 = (y + 17^{\frac{1}{3}})(y^2 - 17^{\frac{1}{3}}y + 17^{\frac{2}{3}})$

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> Method 4 **Factoring out the Greatest Common Divisor**
> A common divisor, found in each term of a polynomial, should be factored and placed in front of the expression, which must now be placed in parentheses. This common divisor might be either 1) a constant, 2) the lowest power of the variable, or 3) a product of both.

$$4x^4 - 28x^2 + 20x - 16 = 4(x^4 - 7x^2 + 5x - 4)$$ **Example 4**

$$7z^2 - 70 = 7(z^2 - 10) = 7(z - \sqrt{10})(z + \sqrt{10})$$

$$x^8 + 4x^4 + 7x^2 = x^2(x^6 + 4x^2 + 7)$$

$$x^5 - 30x^3 = x^3(x^2 - 30) = x^3(x - \sqrt{30})(x + \sqrt{30})$$

$$18x^4 + 24x^9 - 48x^3 - 9x^5 = 3(6x^4 + 8x^9 - 16x^3 - 3x^5) = 3x^3(6x + 8x^6 - 16 - 3x^2)$$

$$5x^{\frac{12}{5}} - 2x^2 + 12x^{\frac{-3}{5}} = x^{\frac{-3}{5}}(5x^3 - 2x^{\frac{13}{5}} + 12)$$

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At this point, before introducing Method 5, we introduce an important fact, or general principle which will help guide us in factoring. We will not discuss why this is true, but simply present it as a guide. It might seem a bit dry or theoretic, and yet at the same time, a very powerful and wide reaching statement about factoring!

> IMPORTANT FACT 1 **The Fundamental Theorem of Algebra**
> Any polynomial (in a single variable) can be factored to the point where all of it's factors are either linear or quadratic

This is very interesting to mathematicians! Throughout the study of mathematics, you will come across these so called **Fundamental Theorems**. As the name suggests, these are the most important facts in any area of mathematics. We will see more of these! In this case, Important Fact 1 is one version of the Fundamental Theorem of Algebra. Many of the Fundamental Theorems have two or three slightly different versions, in order to be specifically applied to lots of different problems!

In theory, the fundamental theorem of algebra promises that **any** polynomial has some set of factors, all of which are either linear or quadratic. However, in practice, it is not always easy, or even possible, to factor a polynomial this completely, using the methods in this chapter.

Example 5 Try factoring the polynomial: $x^4 + 2x^3 - 13x^2 - 14x + 24$, using the rules we have presented above. None of them seem to apply. There is, at this point, no simple way to determine the factors of this polynomial. Are they all linear? Is one or more factor quadratic? Important Fact 1 tells us that there are factors of this polynomial which are either linear or quadratic. Even though we can't **find** these factors, once we know what they are, it is easy to **verify** that they are correct. In particular, we can check that:

$$(x + 4)(x - 3)(x + 2)(x - 1) = x^4 + 2x^3 - 13x^2 - 14x + 24$$

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To make things even more complicated, the coefficients of the factors may not even be rational numbers, even if the coefficients of the original polynomial are! We have already seen several examples of this above, such as:

$$w^2 - 5 \quad = (w - \sqrt{5})(w + \sqrt{5})$$

The truth is, that at this point, we can only give a few rules which can be applied to some very special examples. As a general rule, the lower the degree of the polynomial, the better our chances are of factoring it. There is good news however: First, in actual practice, many of the problems seen in the "real world" assume the form of simple polynomials which can be easily factored by using the rules we present here. Secondly, in future sections, we will show very powerful ways to find the factors of a wide variety of polynomials!

II Method 5: Factoring a Quadratic by Trial and Error

Any quadratic expression can be written as $Ax^2 + Bx + C$ for some real numbers, A, B, and C, and variable x. Using Method 4, if possible, we may always assume that A, B, and C are as simple as possible, and that A is positive. For example:

$$-3x^2 + 2x - 5 \quad = -1(3x^2 - 2x + 5)$$

$$24x^2 + 8x - 4 \quad = 4(6x^2 + 2x - 1)$$

Using Method 4 is not absolutely essential, but does simplify what follows. One might think that such a simple expression should easily factor, using some very simple rules, but this is not always true. However, if a quadratic polynomial can be factored, the result is always a product of two linear factors. This means that if a quadratic expression is factorable at all, it must have the form:

$$Ax^2 + Bx + C = (dx + e)(gx + h) \text{ for some real numbers } d, e, g, h.$$

Our job, then, is to determine the relationship between the constants A, B, C and d, e, g, h. That will be the basis of our trial and error method. Expanding $(dx + e)(gx + h)$, we obtain:

$$\begin{aligned}(dx + e)(gx + h) &= dgx^2 + dhx + egx + eh \\ &= dgx^2 + (dh + eg)x + eh \\ &= Ax^2 + Bx + C\end{aligned}$$

This single relationship gives us 3 equations:

$$A = dg \qquad B = (dh + eg) \qquad C = eh$$

These are easily verified in the following examples:

Example 6

Ax^2	$+$	Bx	$+$	C	$=$	$(dx+e)$	$(gx+h)$
x^2	$+$	$7x$	$+$	10	$=$	$(x+2)$	$(x+5)$
x^2	$-$	$9x$	$+$	20	$=$	$(x-4)$	$(x-5)$
x^2	$+$	$3x$	$-$	18	$=$	$(x+6)$	$(x-3)$
x^2	$-$	$5x$	$-$	14	$=$	$(x+2)$	$(x-7)$
$8x^2$	$-$	$10x$	$-$	3	$=$	$(2x-3)$	$(4x+1)$
$6x^2$	$-$	$10x$	$-$	3	$=$	$3(x-3)$	$(2x+5)$

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Notice how the signs in the factors affect the signs in the product. We can summarize the behavior by listing the four possibilities:

$$Ax^2 + Bx + C \quad = (dx + e)(gx + h) \qquad Ax^2 - Bx + C \quad = (dx - e)(gx - h)$$

$$Ax^2 + Bx - C \quad = (dx + e)(gx - h) \qquad Ax^2 - Bx - C \quad = (dx + e)(gx - h)$$

Let's see how these properties can help factor a quadratic polynomial. We use the four step "trial and error" procedure below:

1. Noting the signs of the quadratic coefficients, set up the parentheses for the two factors, and insert the correct signs:

   Example: $4x^2 - 7x + 3 \quad = \quad (\_\_\_x - \_\_\_)(\_\_\_x - \_\_\_)$

2. Recalling the rule $A = dg$, we guess the two x-coefficients, knowing that their product must equal the leading coefficient of the quadratic polynomial. There will usually be more than one such set which looks reasonable.

   Example: $4x^2 - 7x + 3 \quad = \quad (2x - \_\_\_)(2x - \_\_\_)$
   or $\quad = \quad (4x - \_\_\_)(1x - \_\_\_)$

3. Recalling the rule $C = eh$, we guess the two constant coefficients, knowing that their product must equal the constant coefficient of the quadratic polynomial. There will usually be more than one such set which looks reasonable.

   Example: $4x^2 - 7x + 3 \quad = \quad (2x - 1)(2x - 3)$
   or $\quad = \quad (2x - 3)(2x - 1)$
   or $\quad = \quad (4x - 1)(1x - 3)$
   or $\quad = \quad (4x - 3)(1x - 1)$

4. For each such trial factorization, multiply the factors to determine if the product is, in fact, the quadratic polynomial you started with. If so, you have found your factors. If not, keep looking until you have checked all possibilities

$$\textbf{Example: } (4x-3)(1x-1) \begin{aligned}[t] &= 4x^2-3x-4x+3 \\ &= 4x^2-7x+3\end{aligned}$$

Example 7

Consider the polynomial $5x^2+29x-6$. One way to perform the 4 steps above quickly is to write the potential factorization out similar to:

$$5x^2+29x-6 = (\left\{\begin{array}{c}5\\1\end{array}\right\}x+\left\{\begin{array}{c}6\\3\\2\\1\end{array}\right\})(\left\{\begin{array}{c}1\\5\end{array}\right\}x-\left\{\begin{array}{c}1\\2\\3\\6\end{array}\right\})$$

The above is really just a shorthand notation for steps 1, 2, and 3. To perform Step 4, we must multiply every combination of numbers from $\{5{,}1\}$ with numbers from $\{6, 3, 2, 1\}$. There will be 8 possible products: 30, 15, 10, 5, 6, 3, 2, 1. If one of these 8 products, minus another of these is equal to 29, the middle term of the polynomial, then we have found it's factors! In this case, since $30-1=29$, (or specifically, $5\cdot 6-1\cdot 1=29$), we know that our factorization is:

$$5x^2+29x-6 = (\left\{\begin{array}{c}5\\1\end{array}\right\}x+\left\{\begin{array}{c}6\\3\\2\\1\end{array}\right\})(\left\{\begin{array}{c}1\\5\end{array}\right\}x-\left\{\begin{array}{c}1\\2\\3\\6\end{array}\right\}) = (x+6)(5x-1)$$

which, **of course**, we will verify by expanding $(x+6)(5x-1)$ and seeing that we obtain $5x^2+29x-6$. ♣

A few final words about Method 5. First, we emphasize one more time, that some quadratic polynomials can **not** be factored. Important Fact 1 says that every polynomial can be factored down to the product of linear and quadratic terms, but does not tell us any more than this. Secondly, notice that in steps 2) and 3) above, we are really looking for constant divisors of the coefficients of the leading and constant terms of the quadratic polynomial. Finding these gives us the list of possible factors! Finally, the astute reader has possibly already wondered if steps 1), 2) and 3) above are interchangeable. In fact they are. We can look at the linear coefficients first, or the constant terms first, or place the + or − signs early or later... whichever makes the most sense to the individual.

III Some advanced cases

Below we show some examples which at first seem more complicated. Yet we simply use the same rules above... but a bit more creatively. In particular, we often combine some of the rules above.

Example 8

$v^4 - 625 =$ $(v^2-25)(v^2+25) = (v-5)(v+5)(v^2+25)$
Method 1 used twice

$y^8 - 256 =$ $(y^4-16)(y^4+16) = (y^2-4)(y^2+4)(y^4+16)$
$= (y-2)(y+2)(y^2+4)(y^4+16)$
Method 1 used 3 times

$(3x-5)^2 - 36 =$ $((3x-5)-6)((3x-5)+6) = (3x-11)(3x+1)$
Method 1 applied using the variable $(3x-5)$

$v^4 - 12v^2 + 35 =$ $(v^2-7)(v^2-5) = (v-\sqrt{7})(v+\sqrt{7})(v-\sqrt{5})(v+\sqrt{5})$
Method 5 applied using the variable v^2,
then Method 1 applied twice using the variable v

$w^6 - 117w^3 - 1000 =$ $(w^3-125)(w^3+8) =$
$(w-5)(w^2+5w+25)(w+2)(w^2-2w+4)$
Method 5 applied using the variable w^3,
then Methods 2 and 3 applied to the cubic factors

$r^7b^3 - rb^5 =$ $rb^3(r^6-b^2) = rb^3(r^3-b)(r^3+b)$
$= rb^3(r-b^{\frac{1}{3}})(r^2+rb^{\frac{1}{3}}+b^{\frac{2}{3}})(r+b^{\frac{1}{3}})(r^2-rb^{\frac{1}{3}}+b^{\frac{2}{3}})$
Method 4, then Method 1, then Methods 2 and 3

$40v^3 + 5 =$ $5(8v^3+1) = 5(2v+1)(4v^2-2v+1)$
Method 4, then Method 3

$63v^7 - 28v^3 =$ $7v^3(9v^4-4) = 7v^3(3v^2-2)(3v^2+2)$
$= 7v^3(\sqrt{3}v-\sqrt{2})(\sqrt{3}v+\sqrt{2})(3v^2+2)$
Method 4, then Method 1 twice

$2\frac{m^4}{z^3} - 11\frac{m^3}{z^2} - 6\frac{m^2}{z} =$ $\frac{m^2}{z^3}(2m^2-11mz-6z^2) = \frac{m^2}{z^3}(m-6z)(2m+z)$
Method 4, then Method 5

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IV A Few Higher Degree Factoring Formulas

Look again at Methods 1, 2 and 3. We can give higher order analogs of these formulas. We leave it to the curious or doubting reader to verify that these formulas are correct. Hint: Multiply the factors!

METHOD 5 **The Difference of Positive Integer Powers**
If n is a positive integer, then:

$$x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-3}x^2 + a^{n-2}x + a^{n-1})$$

METHOD 6 **The Sum of Odd Integer Powers**
If n is a positive odd integer, then:

$$x^n + a^n = (x+a)(x^{n-1} - ax^{n-2} + a^2x^{n-3} - \cdots + a^{n-3}x^2 - a^{n-2}x + a^{n-1})$$

Example 9

$$\begin{aligned} x^6 - 64 &= (x-2)(x^5 + 2x^4 + 4x^3 + 8x2 + 16x + 32) \\ x^5 + 8 &= (x + 8^{\frac{1}{5}})(x^4 - 8^{\frac{1}{5}}x^3 + 8^{\frac{2}{5}}x^2 - 8^{\frac{3}{5}}x + 8^{\frac{4}{5}}) \\ x^n - 1 &= (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + ... + x^2 + x + 1) \text{ for any integer n } > 1 \end{aligned}$$

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Example 10

We just verified in the previous example that we can factor $x^6 - 64$ to get:

$$x^6 - 64 = (x-2)(x^5 + 2x^4 + 4x^3 + 8x2 + 16x + 32).$$

Yet, using Method 1, then 2, we get:

$$\begin{aligned} x^6 - 64 &= (x^3 - 8)(x^3 + 8) \\ &= (x-2)(x^2 + 2x + 4)(x+2)(x^2 - 2x + 4) \end{aligned}$$

On the other hand, using Method 2, then 1, we get:

$$\begin{aligned} x^6 - 64 &= (x^2 - 4)(x^4 + 4x^2 + 16) \\ &= (x-2)(x+2)(x^4 + 4x^2 + 16) \end{aligned}$$

Comparing all of these factors, we see that :

$$\begin{aligned}(x^5 + 2x^4 + 4x^3 + 8x2 + 16x + 32) &= (x^2 + 2x + 4)(x + 2)(x^2 - 2x + 4) \\ &= (x + 2)(x^4 + 4x^2 + 16)\end{aligned}$$

which further tells us that:

$$(x^4 + 4x^2 + 16) = (x^2 + 2x + 4)(x^2 - 2x + 4).$$

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The moral to this story: There is often more than one way to factor a polynomial and get different factors. However, there is one and only one set of *prime* factors for any given polynomial, where a prime factor is one that cannot be factored any further. Until we reach a factorization using only prime factors, we often see a vastly different set of factors, depending on which method we use. To see this more clearly, consider an analogous example of factoring an integer, for instance, the number 56. If asked to factor 56, we might say $56 = 8 \cdot 7$, and someone else might say $56 = 14 \cdot 4$. Both of these are true statements and the factors in each case are different. But the astute observer should see that some of the factors can be factored further. Eventually, if we factor as much as possible, we are left with a set of factors which are all prime numbers: $56 = 2 \cdot 2 \cdot 2 \cdot 7$

Exercises for Section 0.1

In problems 1 through 5, factor the polynomials as completely as possible.

1. (a) $v^2 - 100$
   (b) $w^4 - 49$
   (c) $s^3 - 8$
   (d) $x^3 + 64$
   (e) $r^2 + 8$
   (f) $y^3 - 50$
   (g) $3v^3 - 81$
   (h) $5w^5 - 320w^2$
   (i) $11v^2 - 22v + 11$
   (j) $w^4 - 7w^2 - 30$
   (k) $v^3 + 25v^2 + 52$
   (l) $L^{100} - 4L^{99} - 21L^{98}$
   (m) $r^{25} - 6r^{20} + 8r^{15}$
   (n) $(y-3)^2 + 5(y-3) - 14(y-3)$

2. (a) $x^2 - 7x + 12$ (b) $x^2 - 3x - 10$
(c) $x^2 + 4x - 12$ (d) $x^2 + 5x - 24$
(e) $x^2 + 9x + 14$ (f) $x^2 + 10x + 16$
(g) $x^2 + 11x + 24$ (h) $2x^2 + x - 1$
(i) $8x^2 - 6x + 1$ (j) $12x^2 - x - 1$
(k) $4x^2 + 25x + 6$ (l) $6x^2 - 25x + 4$
(m) $x^2 + 30x - 1296$ (n) $x^2 - 6x - 216$

3. (a) $x^4 - 7x^2 + 12$ (b) $x^{18} - 3x^9 - 10$
(c) $(\frac{1}{x})^2 + \frac{4}{x} - 12$ (d) $x^{-4} + 5x^{-2} - 24$
(e) $(5x + 3)^6 + 9(5x + 3)^3 + 14$ (f) $x^5 + 10x^4 + 16x^3$

4. (a) $u^{\frac{7}{3}} - 4u^{\frac{1}{3}}$ (b) $v^{\frac{19}{7}} - v^{\frac{-2}{7}}$
(c) $w^{\frac{33}{2}} + 128w^{\frac{19}{2}}$ (d) $r^{-5} - 8r^{-3}$

5. (a) $x^5 - 1024$ (b) $x^5 - 1022$ (c) $x^8 - 1$
(d) $x^9 - 1$ (e) $x^{128} - 1$ (f) $x^{-8} - 1$

6. Identify as many factors of the polynomial $x^{12} - 4098$ as possible.
Hint: Use as many of the factoring methods as you can.

7. Consider the polynomial $x^4 + 4$.
a) Verify that this quartic polynomial has the following two quadratic factors $(x^2 - 2x + 2)(x^2 + 2x + 2)$
b) What are the factors for the quartic polynomial $x^4 + C$ for any positive number C? Hint: Assume the form $x^4 + C = (x^2 - Rx + S)(x^2 + Rx + S)$, multiply out the factors, and find R and S in terms of C.
c) Factor the polynomial $x^4 + 81$
d) Factor the polynomial $x^4 + 7$

8. Compute the value of $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \frac{1}{729} + \frac{1}{2187} + \frac{1}{6561}$, without actually performing the sum, but using a clever formula.

9. Compute the value of $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \frac{1}{243} + \frac{1}{729} - \frac{1}{2187} + \frac{1}{6561}$, without actually performing the sum, but using a clever formula.